A NEW LINK BETWEEN TEICHMÜLLER THEORY AND COMPLEX DYNAMICS

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Sarah Colleen Koch

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Sarah Colleen Koch, Ph.D.

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Given a Thurston map $f : S^2 \to S^2$ with postcritical set $\mathcal{P}$, C. McMullen proved that the graph of the Thurston pullback map, $\sigma_f : \text{Teich}(S^2, \mathcal{P}) \longrightarrow \text{Teich}(S^2, \mathcal{P})$, covers an algebraic subvariety of $V_f \subset \text{Mod}(S^2, \mathcal{P}) \times \text{Mod}(S^2, \mathcal{P})$. In [2], L. Bartholdi and V. Nekrashevych examined three examples of Thurston maps f, where $|\mathcal{P}| = 4$, identifying $\text{Mod}(S^2, \mathcal{P})$ with $\mathbb{P}^1 - \{0, 1, \infty\}$. They proved that for these three examples, the algebraic subvariety $V_f \subset \mathbb{P}^1 \times \mathbb{P}^1$ is actually the graph of a function $g : \mathbb{P}^1 \to \mathbb{P}^1$ such that $g \circ \pi \circ \sigma_f = \pi$, where $\pi : \text{Teich}(S^2, \mathcal{P}) \longrightarrow \mathbb{P}^1 - \{0, 1, \infty\}$ is the universal covering map. We generalize the Bartholdi-Nekrashevych construction to the case where $|\mathcal{P}|$ is arbitrary and prove that if $f : S^2 \to S^2$ is a Thurston map of degree d whose ramification points are all periodic, then there is a postcritically finite endomorphism $g_f : \mathbb{P}^{|\mathcal{P}|-3} \longrightarrow \mathbb{P}^{|\mathcal{P}|-3}$ such that $g_f \circ \pi \circ \sigma_f = \pi$. Moreover, the complement of the postcritical locus of g_f is Kobayashi hyperbolic.

We prove that if $V_f \subset \mathbb{P}^{|\mathcal{P}|-3} \times \mathbb{P}^{|\mathcal{P}|-3}$ is the graph of such a map g_f, so that the algebraic degree of g_f is d, then g_f is a completely postcritically finite endomorphism. Moreover, we prove in this case that the Thurston pullback map $\sigma_f : \text{Teich}(S^2, \mathcal{P}) \longrightarrow \text{Teich}(S^2, \mathcal{P})$ is a covering map of its image, and it is not surjective. We discuss the dynamics of the maps g_f in the context of Thurston's topological characterization of rational maps, and use the map σ_f to understand the map g_f and vice versa.

BIOGRAPHICAL SKETCH

Sarah Colleen Hanlon Koch was born in Concord, NH. She grew up in Concord, but she spent many wonderful weekends at her grandfather's summer home in Wilmington, Vermont. She attended Shaker Road Child Care Center, Conant Elementary School, and learned that she had a deep appreciation of math and science at Rundlett Junior High School; she was particularly inspired by her eighth grade algebra teacher, Mr. Lemeris. This affinity for math and science continued to grow as she attended Concord High School where her favorite subject was AP Calculus, taught by Ms. Davis.

She started to pursue a chemistry major at Rensselaer Polytechnic Institute but soon realized that the lab requirements interfered with all of the math classes she wanted to take. Encouraged by her undergraduate advisor, Dr. Piper, she subsequently dropped chemistry completely, giving her more time to focus on her favorite subject: mathematics.

In 2002, she enrolled in the mathematics PhD program at Cornell University and started her studies with John H. Hubbard. During most of her tenure as a graduate student, she was supported by a fellowship from the National Physical Science Consortium. Throughout her studies, she was very fortunate to have the opportunities to travel frequently to France, to India, to Mexico, to England, and to Switzerland. She spent 2006–2007 in Marseille, France and earned a Doctorat de Mathématiques from the Université de Provence in May of 2007. She earned her PhD from Cornell in August of 2008.

She plans to spend 2008–2009 at the University of Warwick as a National Science Foundation Postdoctoral Fellow, and then she plans to go to Harvard University as a Benjamin Peirce Assistant Professor, just as her advisor John H. Hubbard did in 1973.

This thesis is dedicated to Mom, Dad, and Lindsey.

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CHAPTER 1

INTRODUCTION

In this thesis we present a systematic way to construct *postcritically finite endomorphisms* of $\mathbb{P}^n$. These endomorphisms arise as maps on a certain moduli space. Three examples of such maps were first constructed by L. Bartholdi and V. Nekrashevych in [2]. We begin by presenting the context in which Bartholdi and Nekrashevych constructed these maps; they arose in the solution of the *twisted rabbit problem.*

1.1 Twisted rabbits

Consider the 'rabbit' polynomial $p_r(z) = z^2+c_r$, where c_r is the 'rabbit' parameter. This polynomial has a critical point at $z_0 = 0$, and a super-attracting cycle of period 3: $0 \mapsto c_r \mapsto c_r^2+c_r \mapsto 0$. The analytic map $p_r : \mathbb{P}^1 \to \mathbb{P}^1$ is a ramified covering map, ramified at 0 and ∞. We mark the *postcritical set* of p_r, $\mathcal{P} := \{\infty, 0, c_r, c_r^2 + c_r\}$ on $\mathbb{P}^1$. Since $|\mathcal{P}| < \infty$, p_r is called a *Thurston map.*

Let γ be a curve on $\mathbb{P}^1 - \mathcal{P}$ separating 0 and ∞ from c_r and $c_r^2 + c_r$, and let D_γ be the Dehn twist around γ. We now consider a new family of Thurston maps $f_n := {D_\gamma}^{\circ n} \circ p_r$, where f_n is called the *n-twisted rabbit map.* For $n > 1$, the map f_n is not analytic; it is merely a ramified cover of S^2, where S^2 is an oriented topological 2-sphere. For each n, $f_n : S^2 \to S^2$ is a *topological polynomial* with the same *ramification portrait* as p_r; that is, each map f_n has simple ramification points at both 0 and ∞, where 0 is periodic of period 3, and ∞ is fixed.

By *Thurston's topological characterization of rational maps*, each $f_n : S^2 \to S^2$ is either *equivalent* to a rational map $F_n : \mathbb{P}^1 \to \mathbb{P}^1$, or is said to be *obstructed.*

A theorem of I. Berstein and S. Levy, (see [26]), asserts that since each f_n is a topological polynomial where all ramification points are periodic, f_n cannot be obstructed. Thus for each n, the map $f_n : S^2 \to S^2$ is equivalent to a rational map $F_n : \mathbb{P}^1 \to \mathbb{P}^1$. Evidently, F_n must have the same ramification portrait as f_n; that is, each F_n must have exactly two simple critical points at 0 and ∞, such that ∞ is fixed and 0 is periodic of period 3. Each F_n is therefore a quadratic polynomial with a super-attracting cycle of period 3.

Up to affine conjugacy, there are exactly three such polynomials: the 'rabbit', the 'corabbit', and the 'airplane' polynomials, denoted as p_r, p_c, and p_a respectively. So for each n, f_n is equivalent to one of them. In the early 1980's, J. H. Hubbard asked which polynomial is equivalent to f_n for an arbitrary n? This is known as the *twisted rabbit problem.*

This problem was solved in 2006 by L. Bartholdi and V. Nekrashevych in [2]. In their paper, they constructed a map

$$g : \mathrm{Mod}(S^2, \mathcal{P}) \dashrightarrow \mathrm{Mod}(S^2, \mathcal{P})$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Teich}(S^2,\mathcal{P}) \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2,\mathcal{P}) \arrow[d, "\pi"] \\
\mathrm{Mod}(S^2,\mathcal{P}) & \mathrm{Mod}(S^2,\mathcal{P}) \arrow[l, dashed, "g"']
\end{tikzcd}$$

where $\mathrm{Teich}(S^2, \mathcal{P})$ is the Teichmüller space, $\mathrm{Mod}(S^2, \mathcal{P})$ is the moduli space,

$$\sigma : \mathrm{Teich}(S^2, \mathcal{P}) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P})$$

is the *Thurston pullback map*, and

$$\pi : \mathrm{Teich}(S^2, \mathcal{P}) \longrightarrow \mathrm{Mod}(S^2, \mathcal{P})$$

is the universal covering map. The map g depends heavily on the postcritical combinatorics of the Thurston map p_r. In [2], Bartholdi and Nekrashevych constructed such a map $g : \mathrm{Mod}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Mod}(S^2, \mathcal{P}_f)$ for two other Thurston maps f, of which the postcritical set $\mathcal{P}_f$ has four elements.

1.2 The unicritical case

In 2007, we generalized this construction in [23] to cases where $|\mathcal{P}_f|$ is arbitrary. The methods in [23] provide a new way to generate *postcritically finite endomorphisms of* $\mathbb{P}^n$. Following the work in [2], these endomorphisms were constructed by using the postcritical combinatorics of a Thurston map f, where f is a *unicritical topological polynomial*.

After identifying $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with an open subset of $\mathbb{P}^n$ where $n = |\mathcal{P}_f| - 3$, we proved that each of the maps $g_f : \mathrm{Mod}(S^2, \mathcal{P}_f) \dashrightarrow \mathrm{Mod}(S^2, \mathcal{P}_f)$ can be extended to postcritically finite endomorphisms $g_f : \mathbb{P}^n \to \mathbb{P}^n$ if the Thurston map f is a topological polynomial which is unicritical, giving us the following commutative diagram.

$$\begin{tikzcd} \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"'] \end{tikzcd}$$

Since the maps g_f were originally found as maps on moduli space, there is a certain amount of Teichmüller theory underlying the construction of g_f. This provides a link between the dynamics of the endomorphisms and the dynamics of the Thurston pullback map. The commutative diagram above allows us to use σ_f to help understand the dynamics of g_f. This link between the two maps raises some very interesting questions. We discuss this in chapter 8.

1.3 Main Results

In this thesis, we prove that the results above extend to larger classes of Thurston maps f. We discuss three different compactifications of $\mathrm{Mod}(S^2, \mathcal{P}_f)$, and the virtues of using the $\mathbb{P}^n$ compactification for the topological polynomials. We then prove that if the Thurston map f is a topological polynomial such that all ramification points of f are periodic, then there exists a *postcritically finite endomorphism* $g_f : \mathbb{P}^n \to \mathbb{P}^n$ such that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"'] \end{tikzcd}$$

We also discuss each of the endomorphisms in the context of a question posed by C. McMullen in 1989. McMullen asked about constructing nontrivial examples of postcritically finite endomorphsims $G : \mathbb{P}^n \to \mathbb{P}^n$ such that the complement of the postcritical locus is Kobayashi hyperbolic. This question was first answered in 1992 in [13]. Fornæss and Sibony constructed two examples of postcritically finite endomorphisms which have this property. Each of the maps constructed in theorem 4.0.1 also has this property.

We then introduce the driving force behind the construction of the endomorphisms, and define the $\pi\sigma$-property of a Thurston map f; this notion captures the key idea behind the construction of the maps g_f. We establish when this induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, and when the constructed maps have points of indeterminacy. We give necessary and sufficient conditions on the portrait of f for an induced map to exist. Under certain hypotheses, we also give necessary and sufficient conditions for the induced map to be holomorphic.

We then discuss how these endomorphisms g_f provide new results into the underlying Teichmüller theory. We prove that if the Thurston map f is a topological polynomial of degree d and there is an endomorphism $g_f : \mathbb{P}^n \to \mathbb{P}^n$, such that the algebraic degree of g_f is equal to d, then the map

$$\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f)$$

is a covering map of its image, and the image is open and dense in $\mathrm{Teich}(S^2, \mathcal{P}_f)$.

In [34], N. Selinger proved that the pullback map σ_f extends to the *augmented Teichmüller space*, which we discuss in chapter 7. The augmented Teichmüller space, $\overline{\mathrm{Teich}(S^2, \mathcal{P}_f)}$, is the topological space obtained when one adds the "surfaces with nodes" boundary to the Teichmüller space (see [1] or [11]). The action of the pure mapping class group extends to the augmented Teichmüller space, and when we quotient by this action, we obtain the *augmented moduli space*, $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}$. In the category of complex analytic spaces, $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}$ is isomorphic to the *Deligne-Mumford compactification* of moduli space, one of the compactifications we discuss in chapter 3. We also address the implications of extending the maps $g_f : \mathrm{Mod}(S^2, \mathcal{P}_f) \dashrightarrow \mathrm{Mod}(S^2, \mathcal{P}_f)$ to the Deligne-Mumford compactification in chapter 7.

We then discuss the forbidden locus, and the stratified structure of the compactified moduli space. We also prove that some of the induced maps are *completely postcritically finite endomorphisms*, a notion that was introduced by Fornæss and Sibony in [13] (see section 7.3.1).

In chapter 8, we define the semi-group $\Theta_{\mathcal{P}}(R)$ for these Thurston maps and use it to classify the periodic cycles of g_f in the moduli space following arguments similar to those in [23].

We then return to the different compactifications of $\mathrm{Mod}(S^2, \mathcal{P}_f)$, and discuss the extensions of the induced maps in chapter 9. For each of the three compactifications introduced in chapter 3, we present an example of a Thurston map for which there is an induced map g_f, which does not extend holomorphically to the compactification.

In general, such a map $g_f : \mathbb{P}^n \to \mathbb{P}^n$ does not exist; instead, one obtains a correspondence on $\mathbb{P}^n \times \mathbb{P}^n$. This is a result of C. McMullen (see proposition 5.1.4 for further discussion). One obvious question to ask is: what are necessary and sufficient conditions on the Thurston map f so that

- there is a *map* $g_f : \mathbb{P}^n \to \mathbb{P}^n$ such that $g_f \circ \pi \circ \sigma_f = \pi$?
- that map is holomorphic on $\mathbb{P}^n$?

We discuss progress on both points above in chapter 10. For the first question above, we introduce *static portraits* and *minimal portraits*, a first step in understanding this problem. This then leads to a detailed discussion of how these endomorphisms depend on the map F_ϕ, which was discussed in chapter 5. For the second question above, under different hypotheses from the theorem in chapter 5, we provide necessary and sufficient conditions for the induced map to be holomorphic.

CHAPTER 2

PRELIMINARIES

2.1 Background

We first establish some notation and standard definitions. All maps in this thesis will be orientation-preserving. We use S^2 to denote an oriented topological 2-sphere, and $\mathbb{P}^n$ to denote n-dimensional complex projective space; specifically, we use $\mathbb{P}^1$ to denote the Riemnn sphere. Let $f : S^2 \to S^2$ be a ramified covering map, and let Ω_f be the set of ramification points of f. This will be called the *critical set* of the map f. According to the Riemann-Hurwitz formula, f has $2d-2$ ramification points, counted with multiplicity. We define the *postcritical set of f* to be

$$\mathcal{P}_f := \bigcup_{n>0} f^{\circ n}(\Omega_f).$$

A *Thurston map* is a ramified covering map $f : S^2 \to S^2$, such that $|\mathcal{P}_f| < \infty$. We suppose for this thesis that $|\mathcal{P}_f| \geq 3$. A Thurston map f is a *topological polynomial* if $\exists\, \omega \in \Omega_f$, such that $f^{-1}(\omega) = \{\omega\}$; we will call this point ∞.

Two Thurston maps $f : S^2 \to S^2$ and $g : S^2 \to S^2$ are *Thurston equivalent* iff there are homeomorphisms $h_0 : (S^2, \mathcal{P}_f) \to (S^2, \mathcal{P}_g)$ and $h_1 : (S^2, \mathcal{P}_f) \to (S^2, \mathcal{P}_g)$ for which $h_0 \circ f = g \circ h_1$ and h_0 is isotopic to h_1 through homeomorphisms which agree on $\mathcal{P}_f$. In particular, we have the following commutative diagram:

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "h_1"] \arrow[d, "f"'] & (S^2, \mathcal{P}_g) \arrow[d, "g"] \\
(S^2, \mathcal{P}_f) \arrow[r, "h_0"] & (S^2, \mathcal{P}_g).
\end{tikzcd}$$

In [10], Douady and Hubbard, following Thurston, give a complete characterization of equivalence classes of rational maps among those of Thurston maps. The characterization takes the following form. A branched covering

$$f : (S^2, \mathcal{P}_f) \to (S^2, \mathcal{P}_f)$$

induces a holomorphic self-map

$$\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \to \mathrm{Teich}(S^2, \mathcal{P}_f)$$

of Teichmüller space (see Section 2 for the definition). Since it is obtained by lifting complex structures under f, we will refer to σ_f as the *pullback map* induced by f. The map f is equivalent to a rational map if and only if the pullback map σ_f has a fixed point. By a generalization of the Schwarz lemma, the Kobayashi metric on a hyperbolic space is not increased by holomorphic maps; by a theorem of Royden in [33], the Teichmüller space $\mathrm{Teich}(S^2, \mathcal{P}_f)$ is Kobayashi hyperbolic, and the Kobayashi metric is the Teichmüller metric on $\mathrm{Teich}(S^2, \mathcal{P}_f)$. Therefore, σ_f does not increase Teichmüller distances. For most maps f, the pullback map σ_f is a contraction, and so a fixed point, if it exists, is unique. We now define the pullback map after reviewing some Teichmüller theory.

2.2 Teichmüller theory

Recall that a Riemann surface is a connected oriented topological surface together with a *complex structure*: a maximal atlas of charts $\phi : U \to \mathbb{C}$ with holomorphic overlap maps. For a given oriented, compact topological surface X, we denote the set of all complex structures on X by $\mathcal{C}(X)$. It is easily verified that an orientation-preserving branched covering map $f : X \to Y$ induces a map $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$;

in particular, for any orientation-preserving homeomorphism $\psi : X \to X$, there is an induced map $\psi^* : \mathcal{C}(X) \to \mathcal{C}(X)$.

Let $A \subset X$ be finite. The Teichmüller space of (X, A) is

$$\mathrm{Teich}(X, A) := \mathcal{C}(X)/\!\sim_A$$

where $c_1 \sim_A c_2$ if and only if $c_1 = \psi^*(c_2)$ for some orientation-preserving homeomorphism $\psi : X \to X$ which is the identity on A, and which is isotopic to the identity relative to A. In view of the homotopy-lifting property, if

- $B \subset Y$ is finite and contains the critical value set Q_f of f, and
- $A \subseteq f^{-1}(B)$,

then $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$ descends to a well-defined map σ_f between the corresponding Teichmüller spaces:

$$\begin{tikzcd}
\mathcal{C}(Y) \arrow[r, "f^*"] \arrow[d] & \mathcal{C}(X) \arrow[d] \\
\mathrm{Teich}(Y, B) \arrow[r, "\sigma_f"] & \mathrm{Teich}(X, A).
\end{tikzcd}$$

This map is known as the *pullback map* induced by f.

In addition if $f : X \to Y$ and $g : Y \to Z$ are orientation-preserving branched covering maps and if $A \subset X$, $B \subset Y$ and $C \subset Z$ are such that

- B contains Q_f and C contains Q_g,
- $A \subseteq f^{-1}(B)$ and $B \subseteq g^{-1}(C)$,

then C contains the critical values of $g \circ f$ and $A \subseteq (g \circ f)^{-1}(C)$. Thus

$$\sigma_{g \circ f} : \mathrm{Teich}(Z, C) \to \mathrm{Teich}(X, A)$$

can be decomposed as $\sigma_{g\circ f} = \sigma_f \circ \sigma_g$:

$$\begin{tikzcd} \mathrm{Teich}(Z,C) \arrow[r, "\sigma_g"] \arrow[rr, bend right, "\sigma_{g\circ f}"'] & \mathrm{Teich}(Y,B) \arrow[r, "\sigma_f"] & \mathrm{Teich}(X,A). \end{tikzcd}$$

For the special case of $\mathrm{Teich}(S^2, \mathcal{P})$, we may use the Uniformization Theorem to obtain the following description. Given a finite set $\mathcal{P} \subset S^2$ we may regard $\mathrm{Teich}(S^2, \mathcal{P})$ as the quotient of the space of all orientation-preserving homeomorphisms $\phi : (S^2, \phi(\mathcal{P})) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}))$ by the equivalence relation $\sim$ whereby $\phi_1 \sim \phi_2$ if there exists a Möbius transformation μ such that $\mu \circ \phi_1 = \phi_2$ on $\mathcal{P}$, and $\mu \circ \phi_1$ is isotopic to ϕ_2 relative to $\mathcal{P}$. The space $\mathrm{Teich}(S^2, \mathcal{P})$ has a natural topology and is complex manifold (see for example [18], and the references therein).

Our *moduli space* $\mathrm{Mod}(S^2, \mathcal{P})$, is the space of all injections $\psi : \mathcal{P} \hookrightarrow \mathbb{P}^1$ modulo postcomposition with Möbius transformations. If ϕ represents an element of $\mathrm{Teich}(S^2, \mathcal{P})$, the restriction $[\phi] \mapsto \phi|_{\mathcal{P}}$ induces a universal covering $\pi : \mathrm{Teich}(S^2, \mathcal{P}) \to \mathrm{Mod}(S^2, \mathcal{P})$ which is a local biholomorphism with respect to the complex structures on $\mathrm{Teich}(S^2, \mathcal{P})$ and $\mathrm{Mod}(S^2, \mathcal{P})$. Note that $\dim(\mathrm{Teich}(S^2, \mathcal{P})) = \dim(\mathrm{Mod}(S^2, \mathcal{P})) = |\mathcal{P}| - 3$.

Let $f : S^2 \to S^2$ be a Thurston map with $|\mathcal{P}_f| \geq 3$. For any $\mathcal{B} \subseteq \mathcal{P}_f$ where $|\mathcal{B}| = 3$, there is an obvious identification of $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with an open subset of $(\mathbb{P}^1)^{\mathcal{P}_f - \mathcal{B}}$. Assume $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$ and let $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ be a homeomorphism representing τ with $\phi|_{\mathcal{B}} = \mathrm{id}|_{\mathcal{B}}$. By the Uniformization Theorem, there exist

- a unique homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$ representing the element $\tau' := \sigma_f(\tau)$ with $\psi|_{\mathcal{B}} = \mathrm{id}|_{\mathcal{B}}$ and
- a unique rational map $F_\phi : (\mathbb{P}^1, \psi(\mathcal{P}_f)) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$,

such that the following diagram commutes.

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & \big(\mathbb{P}^1, \psi(\mathcal{P}_f)\big) \arrow[d, "F_\phi"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi"] & \big(\mathbb{P}^1, \phi(\mathcal{P}_f)\big)
\end{tikzcd}$$

Conversely, if we have such a commutative diagram with F_ϕ holomorphic, then

$$\sigma_f(\tau) = \tau'$$

where $\tau \in \text{Teich}(S^2, \mathcal{P}_f)$ and $\tau' \in \text{Teich}(S^2, \mathcal{P}_f)$ are the equivalence classes of ϕ and ψ respectively.

2.3 Thurston's topological characterization of rational maps

Before stating Thurston's theorem, we require a few more definitions.

Let $f : S^2 \to S^2$ be a Thurston map with postcritical set $\mathcal{P}_f$. A simple closed curve γ is *nonperipheral* if each component of $S^2 - \gamma$ contains no fewer than two points of $\mathcal{P}_f$. A *multicurve*, $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, is a set of simple, closed, disjoint, nonhomotopic, essential, nonperipheral curves in $S^2 - \mathcal{P}_f$. (By essential, we mean not nullhomotopic). A multicurve Γ is *f-stable* if for all $\gamma \in \Gamma$, every nonperipheral component of $f^{-1}(\gamma)$ is homotopic in $S^2 - \mathcal{P}_f$ to a curve in Γ. Following notation in [4], let γ_{ij}^α be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i rel $\mathcal{P}_f$ (where we index the components with α), and let d_{ij}^α be the degree of the map $f|_{\gamma_{ij}^\alpha} : \gamma_{ij}^\alpha \to \gamma_j$. We define the *Thurston linear transformation* $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ as

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{ij}^\alpha} \gamma_i.$$

The matrix of f_Γ has nonnegative entries, so there is a leading eigenvalue which is real and positive by the Perron-Frobenius theorem. Let $\lambda(f_\Gamma)$ denote this leading eigenvalue. The following theorem is *Thurston's topological characterization of rational maps*; a proof of the theorem can be found in [10].

Theorem 2.3.1 (Thurston)**.** *A Thurston map f with hyperbolic orbifold is equivalent to a rational function if and only if for any f-stable multicurve Γ, $\lambda(f_\Gamma) < 1$. In that case, the rational function is unique up to conjugation by a Möbius transformation.*

An f-stable multicurve with $\lambda(f_\Gamma) \geq 1$ is called a *Thurston obstruction.* If the Thurston map f is not equivalent to a rational map, then f is said to be *obstructed.*

2.4 Ramification portraits

We now introduce the main combinatorial object of interest, the *ramification portrait.* Ramification portraits are very similar to *mapping schemes* which were first introduced in [30]. We begin with the definition of mapping scheme taken directly from [5].

Definition 2.4.1. *We say that (V, α, ν) is a* mapping scheme of degree *d if V is a finite set, α is a map form V to V, and ν is a function from V to $\mathbb{N}$, such that the following hold:*

- *Riemann-Hurwitz condition:*

$$\sum_{v \in V} \nu(v) - 1 = 2d - 2.$$

- *Local degree condition:*

$$\textit{for all } w \in V, \sum_{v \,\in\, \alpha^{-1}(w)} \nu(v) \leq d.$$

- *Critical ends condition:*

$$\textit{if } v \in V, \;\textit{ and if } \alpha^{-1}(v) = \emptyset, \;\textit{ then } \nu(v) \geq 2.$$

A ramification portrait is just a mapping scheme where the sets $\alpha(V)$ and $\nu^{-1}(\{n \geq 2\})$ are distinguished.

Definition 2.4.2. *We say that* $R(\Omega, P, \alpha, \nu)$ *is a* ramification portrait of degree d *if*

- Ω *and* P *are finite sets such that* $(\Omega \cup P, \alpha, \nu)$ *is a mapping scheme of degree* d
- $\alpha(\Omega \cup P) = P$
- $\nu^{-1}(\{n \geq 2\}) = \Omega$

We will sometimes use the notation R *for a ramification portrait when there is no ambiguity about* Ω, P, α, ν*.*

It is useful to think of a ramification portrait of degree d as a directed graph with weighted edges where $\Omega \cup P$ is the set of vertices, and there is an edge connecting x_i to x_{i+1} if $x_{i+1} = \alpha(x_i)$. We assign the weight $\nu(x_i)$ to the edge connecting x_i to x_{i+1} as illustrated in the following example of a ramification portrait of degree 2 where $P = \{x_0, x_1, x_2, x_3\}$ and $\Omega = \{x_0, x_3\}$. We label the edges with their weights if and only if the weight is greater than 1.

$$\begin{tikzcd} x_0 \arrow[r, "2"] & x_1 \arrow[r] & x_2 \arrow[ll, bend left] & x_3 \arrow[loop right, "2"] \end{tikzcd}$$

Definition 2.4.3. *Let $R(\Omega, P, \alpha, \nu)$ be a ramification portrait of degree d such that $\Omega \subseteq P$. Then we say that R is* periodic.

Definition 2.4.4. *Let $R(\Omega, P, \alpha, \nu)$ be a ramification portrait of degree d such that $\Omega \not\subseteq P$. Then we say that R is* preperiodic.

Directly following the treatment of mapping schemes in [5], we present analogous definitions for ramification portraits.

Definition 2.4.5. *Let $R_1(\Omega_1, P_1, \alpha_1, \nu_1)$ and $R_2(\Omega_2, P_2, \alpha_2, \nu_2)$ be ramification portraits of degree d. Then R_1 and R_2 are* isomorphic *if there is a bijection*

$$\beta : \Omega_1 \cup P_1 \to \Omega_2 \cup P_2$$

such that the following two diagrams commute,

$$\begin{tikzcd} \Omega_1 \cup P_1 \arrow[r, "\beta"] \arrow[d, "\alpha_1"'] & \Omega_2 \cup P_2 \arrow[d, "\alpha_2"] \\ P_1 \arrow[r, "\beta|_{P_1}"] & P_2 \end{tikzcd} \qquad \begin{tikzcd} \Omega_1 \cup P_1 \arrow[rr, "\beta"] \arrow[dr, "\nu_1"'] & & \Omega_2 \cup P_2 \arrow[dl, "\nu_2"] \\ & \mathbb{N} & \end{tikzcd}$$

and we write $R_1 \sim_{iso} R_2$. Any such map β is called an isomorphism *between R_1 and R_2.*

Intuitively, R_1 and R_2 are isomorphic if, when thought of as directed graphs, R_1 and R_2 are isomorphic. The relation $\sim_{\text{iso}}$ is evidently an equivalence relation on the set of ramification portraits.

Definition 2.4.6. *Let f be a Thurston map. Then the* ramification portrait of f *is the ramification portrait $R_f(\Omega_f, \mathcal{P}_f, f, \text{loc deg} f)$; it is a ramification portrait of degree $\deg(f)$.*

Definition 2.4.7. *Let f be a Thurston map, and let $R(\Omega, P, \alpha, \nu)$ be a ramification portrait of degree d. We say that f* realizes *R if R and R_f are isomorphic.*

Definition 2.4.8. *A ramificaiton portrait $R(\Omega, P, \alpha, \nu)$ of degree d is of* polynomial type *if there exists a $\omega \in \Omega \cap P$ such that $\alpha(\omega) = \omega$ and $\nu(\omega) = d$.*

Definition 2.4.9. *Let $R(\Omega, P, \alpha, \nu)$ be a ramification portrait of degree d of polynomial type. Then R is* unicritical *if $|\Omega| = 2$.*

One may be tempted to wonder if every ramification portrait of degree d is realizable by some Thurston map f. In theorem 2.1 of [5], the authors exhibit a mapping scheme such that a ramification portrait consistent with this mapping scheme cannot be realized by any ramified covering $f : S^2 \to S^2$.

As mentioned in [5], a result of Thom implies the following theorem.

Theorem 2.4.1. *Let $R(\Omega, P, \alpha, \nu)$ be a ramification portrait of degree d of polynomial type. Then there exists a Thurston map f which realizes R.*

Observe that a Thurston map f realizes a ramification portrait of degree d of polynomial type if and only if f is conjugate to a topological polynomial p; that is, $f = h \circ p \circ h^{-1}$, where $h : S^2 \to S^2$ is a homeomorphism.

We conclude this section with two examples of ramification portraits.

2.4.1 Examples

Example 2.4.1. If $f(z) = z^2 + i$, then R_f is represented by:

$$\begin{tikzcd} 0 \arrow[r, "2"] & i \arrow[r] & -1+i \arrow[r] & -i \arrow[l, bend left] & & \infty \arrow[loop right, "2"] \end{tikzcd}$$

Example 2.4.2. Let $\Omega = \{\omega_0, \omega_1, x_3\}$, $P = \{x_0, x_1, x_2, x_3\}$ then

$$\begin{tikzcd} \omega_0 \arrow[r, "3"] & x_1 \arrow[r] & x_2 \arrow[r] & x_3 \arrow[loop right, "2"] \\ & & \omega_1 \arrow[u, "2"] & \end{tikzcd}$$

represents a preperiodic ramification portrait R of degree 3.

2.5 Postcritically finite endomorphsims

A map $G : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism if it is holomorphic; in particular, it has no points of indeterminacy.

Let $G : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism, and let C_1 be the *critical locus of* G; that is, the set where the Jacobian vanishes. This set is algebraic of codimension 1. We define the *postcritical locus of* G to be

$$D_1 := \bigcup_{n>0} G^{\circ n}(C_1).$$

Definition 2.5.1. *The map G is a* postcritically finite endomorphism *if D_1 is algebraic.*

The set D_1 is algebraic if and only if each component of C_1 is preperiodic under G.

CHAPTER 3

COMPACTIFICATIONS OF THE MODULI SPACE

We begin with some standing assumptions. Suppose that $\mathcal{P} \subset S^2$, such that $|\mathcal{P}| = n+3$, for a fixed $n \geq 0$, and suppose that $\mathcal{P} = \{p_1, \ldots, p_{n+3}\}$. We now explore some of the different compactifications of $\mathrm{Mod}(S^2, \mathcal{P})$. Recall that $\mathrm{Mod}(S^2, \mathcal{P})$ is the set of all injective maps $\phi : \mathcal{P} \hookrightarrow \mathbb{P}^1$ modulo postcomposition by Möbius transformations.

3.1 The $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ compactification

Let $\phi \in \mathrm{Mod}(S^2, \mathcal{P})$, and let a, b, c be three distinct points of $\mathbb{P}^1$. For some $i, j, k \in [1, n+3]$, let $\mu^{a,b,c}_{i,j,k} : \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transormation such that $\mu^{a,b,c}_{i,j,k}(\phi(p_i)) = a, \mu^{a,b,c}_{i,j,k}(\phi(p_j)) = b$, and $\mu^{a,b,c}_{i,j,k}(\phi(p_k)) = c$. Then the element ϕ is specified by where the remaining points of $\mathcal{P}$ are mapped under $\mu^{a,b,c}_{i,j,k} \circ \phi$. For each $m \in \{1, \ldots, n+3\} - \{i, j, k\}$ define $z_m := \mu^{a,b,c}_{i,j,k}(\phi(p_m))$. Order the n points z_m by their indices $(z_{m_1}, z_{m_2}, \ldots, z_{m_n})$, and re-index to obtain the element $(z_1, z_2, \ldots, z_n) \in \mathbb{P}^1 - \{a, b, c\} \times \cdots \times \mathbb{P}^1 - \{a, b, c\}$. In this way, we identify $\mathrm{Mod}(S^2, \mathcal{P})$ with an open subset of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ where there are n copies of $\mathbb{P}^1$, and all coordinates are distinct from each other, and from a, b, and c:

$$\mathrm{Mod}(S^2, \mathcal{P}) \approx \prod_{i=1}^{n} \left(\mathbb{P}^1 - \{a, b, c\}\right) - \Upsilon$$

where $\Upsilon := \{z_i = z_j \text{ where } 1 \leq i < j \leq n\}$. The universal cover

$$\pi : \mathrm{Teich}(S^2, \mathcal{P}) \longrightarrow \mathrm{Mod}(S^2, \mathcal{P})$$

is then identified with the universal cover

$$\pi : \mathrm{Teich}(S^2, \mathcal{P}) \longrightarrow \prod_{i=1}^{n} \left(\mathbb{P}^1 - \{a, b, c\}\right) - \Upsilon.$$

With this identification, the compactification of $\mathrm{Mod}(S^2, \mathcal{P})$ is the product of $\mathbb{P}^1$:

$$\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\prod \mathbb{P}^1} = \prod_{i=1}^{n} \mathbb{P}^1.$$

This compactification is very symmetric in the sense that a, b, and c can be any three distinct points of $\mathbb{P}^1$.

3.1.1 Changing the normalization

It is natural to wonder how the compactification depends on the choice of the points $a, b, c \in \mathbb{P}^1$; we can postcompose with some other Möbius transformation ν, and this will effectively induce an automorphism of the compactification, acting on each factor of $\mathbb{P}^1$:

$$\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\prod \mathbb{P}^1} = \prod_{i=1}^{n} \mathbb{P}^1 = \prod_{i=1}^{n} \nu\left(\mathbb{P}^1\right).$$

This is very natural, but as we will see, this is not the case for all compactifications.

3.2 The $\mathbb{P}^n$ compactification

We now consider the special case of the above where either a, b or c is equal to ∞. Without loss of generality, suppose that $c = \infty$. As above, we then identify $\mathrm{Mod}(S^2, \mathcal{P})$ with an open subset of $\mathbb{P}^1 - \{a, b, \infty\} \times \cdots \times \mathbb{P}^1 - \{a, b, \infty\}$ where the coordinates are distinct. At this point, we can now go further and identify $\mathrm{Mod}(S^2, \mathcal{P})$ with an open subset of $\mathbb{C} - \{a, b\} \times \cdots \times \mathbb{C} - \{a, b\}$ where the coordinates are distinct. Consider the map

$$\Psi : \prod_{i=1}^{n} (\mathbb{C} - \{a, b\}) - \Upsilon \longrightarrow \mathbb{P}^n - \Delta_{a,b} \text{ defined by}$$

$$\Psi : (z_1, z_2, \ldots, z_n) \longmapsto [z_1 : z_2 : \ldots : z_n : 1],$$

where $\Delta_{a,b}$ is the set of $[z_1 : \ldots : z_{n+1}] \in \mathbb{P}^n$ such that at least one of the following holds:

- $\exists\, i \in [1, n]$ with $z_i = a z_{n+1}$
- $\exists\, i \in [1, n]$ with $z_i = b z_{n+1}$
- $\exists\, i, j \in [1, n]$, $i \neq j$ with $z_i = z_j$
- $z_{n+1} = 0$.

Remark 3.2.1. Note that $\Delta_{a,b} = \Delta_{b,a}$.

The map Ψ is an isomorphism, so we have effectively identified $\mathrm{Mod}(S^2, \mathcal{P})$ with an open subset of $\mathbb{P}^n$:

$$\mathrm{Mod}(S^2, \mathcal{P}) \approx \mathbb{P}^n - \Delta_{a,b},$$

and we identify the universal cover

$$\pi : \mathrm{Teich}(S^2, \mathcal{P}) \longrightarrow \mathrm{Mod}(S^2, \mathcal{P})$$

with the universal cover

$$\pi : \mathrm{Teich}(S^2, \mathcal{P}) \longrightarrow \mathbb{P}^n - \Delta_{a,b}.$$

We compactify $\mathrm{Mod}(S^2, \mathcal{P})$ as

$$\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathbb{P}^n} = \mathbb{P}^n.$$

Definition 3.2.1. *The set $\Delta_{a,b} \subset \mathbb{P}^n$ defined above is called* the forbidden locus *of the compactification.*

Proposition 3.2.1. *The forbidden locus $\Delta_{a,b}$ is a union of*

$$\frac{(n+2)(n+1)}{2}$$

hyperplanes.

Proof. We prove this by counting: the first two bullets in definition 3.2.1 give $n+n$ or $2n$ hyperplanes in $\Delta_{a,b}$. The third bullet point above gives n choose 2, or $n!/((n-2)!2!)$ hyperplanes, and the last bullet in definition 3.2.1 gives 1 hyperplane, so in total, we have:

$$2n + \frac{n!}{(n-2)!2!} + 1 = 2n + \frac{n(n-1)}{2} + 1 = \frac{4n+2+n^2-n}{2} = \frac{(n+1)(n+2)}{2}.$$

$\square$

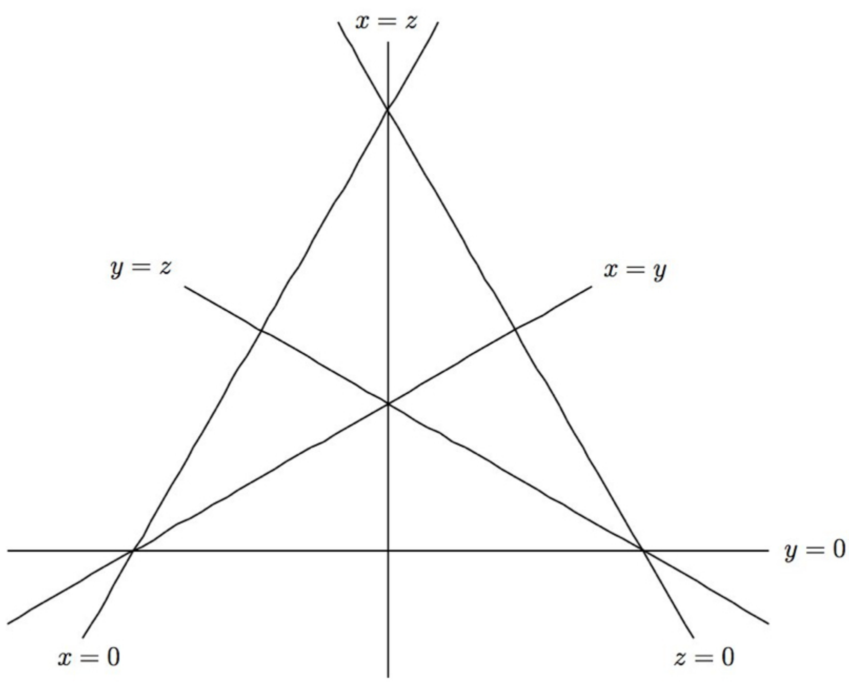


Figure 3.1: The forbidden locus $\Delta_{0,1}$ in $\mathbb{P}^2$.

3.2.1 Changing the normalization

In summary, we have identified $\mathrm{Mod}(S^2, \mathcal{P})$ with

$$\prod_{i=1}^{n} \left(\mathbb{P}^1 - \{a, b, \infty\}\right) - \Upsilon \approx \prod_{i=1}^{n} (\mathbb{C} - \{a, b\}) - \Upsilon \approx \mathbb{P}^n - \Delta_{a,b}.$$

As in section 3.1.1, we can postcompose with some Möbius transformation ν, and see how our compactification is affected. We previously mentioned that in order to use the $\mathbb{P}^n$ compactification of $\mathrm{Mod}(S^2, \mathcal{P})$, it is necessary that ∞ is one of the points a, b, or c. We supposed for section 3.2 that $c = \infty$. If none of $\nu(a), \nu(b)$, or $\nu(\infty)$ is equal to ∞, then this construction fails, and $\mathbb{P}^n$ is not an admissible compactification of $\mathrm{Mod}(S^2, \mathcal{P})$. We therefore suppose that $\nu(\{a, b, \infty\}) = \{s, t, \infty\}$, and we identify $\mathrm{Mod}(S^2, \mathcal{P})$ with

$$\prod_{i=1}^{n} \left(\mathbb{P}^1 - \{s, t, \infty\}\right) - \Upsilon = \prod_{i=1}^{n} (\mathbb{C} - \{s, t\}) - \Upsilon \approx \mathbb{P}^n - \Delta_{s,t}.$$

Postcomposing with ν induces a map from $\mathcal{V} : \mathbb{P}^n - \Delta_{a,b} \to \mathbb{P}^n - \Delta_{s,t}$ given by

$$\mathcal{V} : [z_1 : \ldots : z_n : 1] \longmapsto [\nu(z_1) : \ldots : \nu(z_n) : 1].$$

This map does *not necessarily* extend to the respective compactifications as we now see.

Proposition 3.2.2. *The map $\mathcal{V}$ extends to an automorphism $\mathcal{V} : \mathbb{P}^n \to \mathbb{P}^n$ if and only if $\nu(\infty) = \infty$.*

Proof. Suppose first that $\nu(\infty) = \infty$, so that $\nu(x) = \alpha x + \beta$. We extend the mapping $\mathcal{V} : \mathbb{P}^n - \Delta_{a,b} \to \mathbb{P}^n - \Delta_{s,t}$ with its formula in homogeneous coordinates:

$$\mathcal{V} : [z_1 : \ldots : z_i : \ldots : z_{n+1}] \longmapsto [\alpha z_1 + \beta z_{n+1} : \ldots : \alpha z_i + \beta z_{n+1} : \ldots : z_{n+1}].$$

This is evidently an automorphism of $\mathbb{P}^n$, where $\mathcal{V}(\Delta_{a,b}) = \Delta_{s,t}$.

Suppose now that $\nu(x) = (\alpha x + \beta)/(\delta x + \rho)$, where $\delta \neq 0$. We again extend the mapping $\mathcal{V} : \mathbb{P}^n - \Delta_{a,b} \to \mathbb{P}^n - \Delta_{s,t}$ with its formula in homogeneous coordinates:

$$\mathcal{V} : [z_1 : \ldots : z_i : \ldots : z_{n+1}] \longmapsto$$

$$\left[(\alpha z_1 + \beta z_{n+1})(z_{n+1})^{n-1} : \ldots : (\alpha z_i + \beta z_{n+1})(z_{n+1})^{n-1} : \ldots : \prod_{j=1}^{n} (\delta z_j + \rho z_{n+1})\right]$$

which is not even an endomorphism of $\mathbb{P}^n$: there is are points of indeterminacy at each element of $\mathcal{I} := \{[z_1 : \ldots : z_n : 0] \in \mathbb{P}^n \mid \exists\, i \in [1, n] \text{ with } z_i = 0\}$. $\square$

Remark 3.2.2. We will denote $\Delta_{0,1}$ as Δ.

Remark 3.2.3. It is important to note that the identification of $\mathrm{Mod}(S^2, \mathcal{P})$ with an open subset of $\mathbb{P}^n$ is not symmetric in the sense that it requires ∞ to be one of the points a, b, or c. So this compactification has a preference, or bias for ∞; hence, this compactification is more natural for cases where f is a topological polynomial since ∞ is special for these Thurston maps.

3.2.2 The standard identification of $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$

General Thurston maps

We establish our standard normalization here. Let $f : S^2 \to S^2$ be a Thurston map of degree d, with postcritical set $\mathcal{P}_f = \{p_1, \ldots, p_{n+3}\}$. We identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the following way. Let $\delta \in \mathrm{Mod}(S^2, \mathcal{P}_f)$, and suppose that $\delta(p_{n+1}) = 1, \delta(p_{n+2}) = 0$, and $\delta(p_{n+3}) = \infty$. Define $x_i := \delta(p_i)$ for $i \in [1, n]$. Then we naturally identify δ with the point $[x_1 : \ldots : x_n : 1] \in \mathbb{P}^n$, and we will say that $\mathrm{Mod}(S^2, \mathcal{P}_f)$ is identified with $\mathbb{P}^n - \Delta$ via the *standard identification.*

Let $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ be a homeomorphism, normalized so that $\phi(p_{n+1}) = 1, \phi(p_{n+2}) = 0$, and $\phi(p_{n+3}) = \infty$. Define $x_i := \phi(p_i)$ for all $i \in [1, n]$. We will say that ϕ is a *normalized homeomorphism* if it is normalized in this particular way.

Topological polynomials

For most of this thesis, our Thurston maps f will be topological polynomials with postcritical sets $\mathcal{P}_f$, of which ∞ is a distinguished element. Hence the $\mathbb{P}^n$ compactification of $\mathrm{Mod}(S^2, \mathcal{P}_f)$ is most natural for our calculations.

For the special case where f is a topological polynomial, the standard identification is as follows: we enumerate the postcritical set as $\mathcal{P}_f = \{p_1, \ldots, p_{n+2}, \infty\}$, where $\delta(p_{n+1}) = 1, \delta(p_{n+2}) = 0$, and $\delta(\infty) = \infty$, and define $x_i := \delta(p_i)$ for $i \in [1, n]$. This is the *standard identification* of $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the case where f is a topological polynomial.

If f is a topological polynomial, the homeomorphism

$$\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$$

is a *normalized homeomorphism* if $\phi(p_{n+1}) = 1, \phi(p_{n+2}) = 0$, and $\phi(\infty) = \infty$, and we define $x_i := \phi(p_i)$ for $i \in [1, n]$.

3.3 The Deligne-Mumford compactification

The last compactification of $\mathrm{Mod}(S^2, \mathcal{P})$ we consider is the Deligne-Mumford compactification, whose definition requires a bit of algebraic geometry. Because of its rather abstract definition, this compactification is less accessible than the other compactifications discussed in this chapter. We first review some necessary background.

3.3.1 Preliminaries

We denote the category of sets, the category of complex manifolds, and the category of complex spaces as **Sets**, **ComplexManifolds**, and **ComplexSpaces** respectively. The following definitions are taken from [25].

Definition 3.3.1. *A category C is* locally small *if for every pair of objects A, B,* $\mathrm{Hom}(A, B)$ *is a set.*

Definition 3.3.2. *Let C be a locally small category. For each object A of C, let* $\mathrm{Hom}(\bullet, A)$ *be the contravariant functor which maps objects $X \in C$ to the set* $\mathrm{Hom}(X, A)$*. A contravariant functor $F : C \to$ **Sets** is said to be* representable *if it is naturally isomorphic to* $\mathrm{Hom}(\bullet, A)$ *for some object A of C. A representation of F is a pair (A, Φ) where*

$$\Phi : \mathrm{Hom}(\bullet, A) \to F$$

is a natural isomorphism.

Representations of functors are unique up to unique isomorphism. That is if (A_1, Φ_1) and (A_2, Φ_2) represent the same functor, then there exists a unique isomorphism

$$\phi : A_1 \to A_2$$

such that

$$\Phi_1^{-1} \circ \Phi_2 = \mathrm{Hom}(\bullet, \phi)$$

as natural isomorphisms from $\mathrm{Hom}(\bullet, A_2)$ to $\mathrm{Hom}(\bullet, A_1)$.

3.3.2 The construction

The following is adapted from [16].

Definition 3.3.3. *A* stable $(n+3)$-pointed curve *is a complete connected curve* $\mathcal{C}$ *that has only nodes as singularities, together with an ordered collection* $p_1, \ldots, p_{n+3} \in \mathcal{C}$ *of distinct smooth points of* $\mathcal{C}$*, such that the* $(n+4)$*-tuple* $(\mathcal{C}; p_1, \ldots, p_{n+3})$ *has only finitely many automorphisms.*

Let $\mathbf{SC}_{g,n+3} : \mathbf{ComplexSpaces} \to \mathbf{Sets}$ be a functor defined as follows:
$\mathbf{SC} : T \mapsto \{$(pure) isomorphism classes of stable $(n+3)$-pointed curves X_g of genus g over T together with sections $s_1, \ldots, s_{n+3} : T \to X$ with disjoint images such that the images lie in the smooth subset of $X_g\}$.

Definition 3.3.4. *The functor* $\mathbf{SC}_{g,n+3}$ *defined above is called the* pure $(g, n+3)$-moduli functor.

Definition 3.3.5. *If* $\mathbf{SC}_{g,n+3}$ *is representable by some complex space,* $\mathcal{M}_{g,n+3}$*, then* $\mathcal{M}_{g,n+3}$ *is said to be* the fine moduli space *for* $\mathbf{SC}_{g,n+3}$.

Proposition 3.3.1. *The moduli functor defined above is representable if* $g = 0$*; in fact,* $\mathbf{SC}_{0,n+3}$ *is representable in the category of complex manifolds. That is,*

$$\mathbf{SC}_{0,n+3} : \mathbf{ComplexManifolds} \longrightarrow \mathbf{Sets}$$

is representable. We denote the complex manifold representing $\mathbf{SC}_{0,n+3}$ *as* $\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}}$.

For the proof of the preceeding proposition, see [16].

Definition 3.3.6. *The* Deligne-Mumford compactification of $\mathrm{Mod}(S^2, \mathcal{P})$ *is the fine moduli space* $\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}}$.

If X_g is an oriented, compact topological surface of genus g with marked points in the set A, then the Deligne-Mumford compactification of $\mathrm{Mod}(X_g, A)$ is much more complicated; in that case, $\mathbf{SC}_{g,n+3}$: **ComplexSpaces** $\longrightarrow$ **Sets** is often not representable, so $\overline{\mathrm{Mod}(X_g, A)}_{\mathrm{DM}}$ is called a *coarse moduli space*, and in particular, it is not a manifold. For more information on the general case, see [16]. However, $\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}}$ is a little easier to understand; for one thing, it is a compact complex manifold. The following is extracted from [27].

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n+2}\}$ denote a set of $n+2$ points in $\mathbb{P}^n$ in general position.

Definition 3.3.7. *For* $d \in [1, n]$*, let* $\{\alpha_1, \ldots, \alpha_d\} \subset \{1, \ldots, n+2\}$*, and let* $\Pi_{\alpha_1,\ldots,\alpha_d}$ *denote the span of the points* $\{\mathbf{x}_{\alpha_1}, \ldots, \mathbf{x}_{\alpha_d}\}$*. These are the* Π*-planes of* $\mathbb{P}^n$.

Definition 3.3.8. *The space* $\widehat{\mathbb{P}^n}^{\Delta}$ *is the* sequential blow up space of $\mathbb{P}^n$ *on all* Π*-planes starting with those of lowest dimension and increasing.*

The space $\widehat{\mathbb{P}^n}^{\Delta}$ is therefore obtained by first blowing up the $n+2$ points in general position, and then the *proper transforms* of the lines between the pairs of points, and so on. In [27], A. Lloyd-Philipps proves the following theorem.

Theorem 3.3.1. *The Deligne-Mumford compactification,* $\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}}$ *is isomorphic to* $\widehat{\mathbb{P}^n}^{\Delta}$ *in the category of complex manifolds.*

The cases where $|\mathcal{P}| = 3$ and $|\mathcal{P}| = 4$

If $|\mathcal{P}| = 3$ or $|\mathcal{P}| = 4$, then all three compactifications coincide, and we have

- $|\mathcal{P}| = 3$:

$$\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\prod \mathbb{P}^1} = \overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathbb{P}^n} = \overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}} = \{\text{a point}\}$$

- $|\mathcal{P}| = 4$:

$$\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\prod \mathbb{P}^1} = \overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathbb{P}^n} = \overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}} = \mathbb{P}^1.$$

The case where $|\mathcal{P}| = 5$

Suppose that $|\mathcal{P}| = 5$, so that $n = 2$. As discussed above,

$$\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\prod \mathbb{P}^1} = \mathbb{P}^1 \times \mathbb{P}^1 \qquad \text{and} \qquad \overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathbb{P}^n} = \mathbb{P}^2.$$

According to theorem 3.3.1, $\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}}$ is isomorphic to $\mathbb{P}^2$ blown up at four points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ in general position. We will return to the discussion of this space in chapter 9.

3.3.3 Changing the normalization

The virtue of the Deligne-Mumford compactification is that it is a completely symmetric compactification; the construction is entirely independent of normalization. We never specified any normalization at any point in the discussion of this compactification. From this point if view, this compactification is very natural, however, it is a little more inaccessible than some of the others presented.

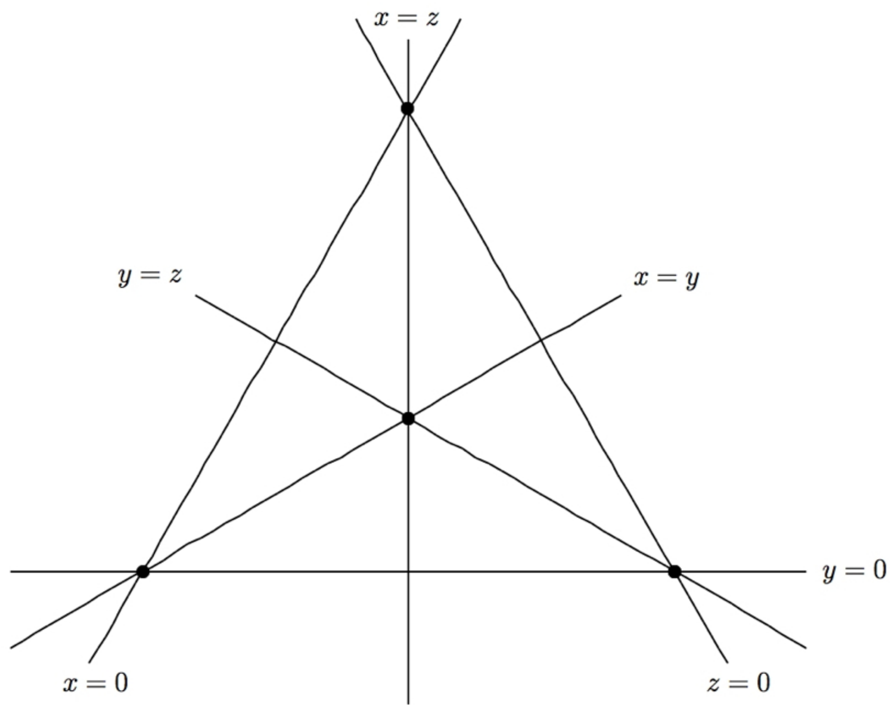


Figure 3.2: If $|\mathcal{P}| = 5$, the Deligne-Mumford compactification $\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}}$ is obtained by blowing up $\mathbb{P}^2$ at the four points of triple intersection in Δ: $[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0], [1 : 1 : 1]$, which are marked above.

3.4 General Thurston maps

As previously mentioned in section 3.2.2, if the Thurston map f is a topological polynomial, then the $\mathbb{P}^n$ compactification of $\mathrm{Mod}(S^2, \mathcal{P}_f)$ is natural. However, if f is just a general Thurston map without a distinguished point $\infty \in \mathcal{P}_f$, then it is unclear which compactification to use. In fact, in this case, the $\mathbb{P}^n$ compactification is decidedly not natural since there is no distinguished point in the postcritical set. We will discuss some examples of such maps in chapter 9, and consider each of the three compactifications separately, emphasizing some of the key differences.

CHAPTER 4

POSTCRITICALLY FINITE ENDOMORPHISMS

Generalizing a result of Bartholdi and Nekrashevych [2], we showed in [23] that if $f : S^2 \to S^2$ is a unicritical topological polynomial with postcritical set $\mathcal{P}_f$, then there is a postcritically finite endomorphism $g_f : \mathbb{P}^n \to \mathbb{P}^n$ for which the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"']
\end{tikzcd}$$

We now show that a similar result holds when f is a topological polynomial whose ramification points are all periodic.

Theorem 4.0.1. *Let $R(\Omega, P, \alpha, \nu)$ be a ramification portrait of degree d, of polynomial type which is periodic. Let f be any topological polynomial with postcritical set $\mathcal{P}_f$, realizing R. Identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ as detailed below. There exists a postcritically finite endomorphism $g_f : \mathbb{P}^n \to \mathbb{P}^n$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"']
\end{tikzcd}$$

where $n := |\mathcal{P}_f| - 3$.

Proof. We proceed with the proof in three steps: we first construct $g_f : \mathbb{P}^n \to \mathbb{P}^n$, and prove that it is an endomorphism of $\mathbb{P}^n$ which makes the diagram commute, we then prove that the critical locus of g_f is contained in the forbidden locus Δ, and lastly, we prove that $g_f(\Delta) \subseteq \Delta$, which will prove that the map g_f is postcritically finite.

Let $R(\Omega, P, \alpha, \nu)$ be a ramification portrait of degree d, of polynomial type which is periodic. Let f be any Thurston map which realizes R. Then there is an isomorphism $\beta : \Omega \cup P \to \Omega_f \cup \mathcal{P}_f$ for which the following two diagrams commute.

$$\begin{tikzcd} \Omega \cup P \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \Omega_f \cup \mathcal{P}_f \arrow[d, "f"] \\ P \arrow[r, "\beta|_P"] & \mathcal{P}_f \end{tikzcd} \qquad \begin{tikzcd} \Omega \cup P \arrow[rr, "\beta"] \arrow[dr, "\nu"'] & & \Omega_f \cup \mathcal{P}_f \arrow[dl, "\text{loc deg } f"] \\ & \mathbb{N} & \end{tikzcd}$$

Enumerate the points of P; $P = \{q_0, q_1, \ldots, q_{n+2}\}$. Then we write the postcritical points in $\mathcal{P}_f$ as $p_i := \beta(q_i)$, so $\mathcal{P}_f = \{p_0, p_1, \ldots, p_{n+2}\}$. Since f is a topological polynomial, there is a $p_i = \infty$, say $p_{n+2} = \infty$. Because R is periodic, $\Omega \subseteq P$, and so $\Omega_f \subseteq \mathcal{P}_f$. Therefore

$$f|_{\mathcal{P}_f} : \mathcal{P}_f \to \mathcal{P}_f$$

is a bijection which induces a permutation fixing ∞. Let $\mu : [0, n+1] \to [0, n+1]$ be the permutation defined by:

$$p_{\mu(k)} = f(p_k)$$

and denote by ν the inverse of μ. We will exploit the fact that f restricted to $\mathcal{P}_f$ is a permutation, for our subsequent calculations.

For $k \in [0, n+1]$, let m_k be the multiplicity of p_k as a critical point of f (if p_k is not a critical point of f, then $m_k := 0$).

Let $n = |\mathcal{P}_f| - 3$. We will identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with an open subset of $\mathbb{P}^n$ as follows. Any point of $\mathrm{Mod}(S^2, \mathcal{P}_f)$ has a representative $\psi : \mathcal{P}_f \hookrightarrow \mathbb{P}^1$ such that

$$\psi(\infty) = \infty \quad \text{and} \quad \psi(p_0) = 0.$$

Two such representatives are equal up to multiplication by a nonzero complex number. We identify the point in $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with the point

$$[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n \quad \text{where} \quad x_1 := \psi(p_1) \in \mathbb{C}, \ldots, x_{n+1} := \psi(p_{n+1}) \in \mathbb{C}.$$

In this way, the moduli space $\mathrm{Mod}(S^2, \mathcal{P}_f)$ is identified with $\mathbb{P}^n - \Delta$.

Set $a_0 := 0$ and let $Q \in \mathbb{C}[a_1, \ldots, a_{n+1}, z]$ be the homogeneous polynomial of degree d defined by

$$Q(a_1, \ldots, a_{n+1}, z) := \int_{a_{\nu(0)}}^{z} \left(d \prod_{k=0}^{n+1} (w - a_k)^{m_k} \right) \mathrm{d}w.$$

Given $\mathbf{a} \in \mathbb{C}^{n+1}$, let $F_{\mathbf{a}} \in \mathbb{C}[z]$ be the monic polynomial defined by

$$F_{\mathbf{a}}(z) := Q(a_1, \ldots, a_{n+1}, z).$$

Note that $F_{\mathbf{a}}$ is the unique monic polynomial of degree d which vanishes at $a_{\nu(0)}$ and whose critical points are exactly those points a_k for which $m_k > 0$, counted with multiplicity m_k.

Let $G_f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be the homogeneous map of degree d defined by

$$G_f \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} := \begin{pmatrix} F_{\mathbf{a}}(a_{\nu(1)}) \\ \vdots \\ F_{\mathbf{a}}(a_{\nu(n+1)}) \end{pmatrix} = \begin{pmatrix} Q(a_1, \ldots, a_{n+1}, a_{\nu(1)}) \\ \vdots \\ Q(a_1, \ldots, a_{n+1}, a_{\nu(n+1)}) \end{pmatrix}.$$

We claim that $G_f^{-1}(\mathbf{0}) = \{\mathbf{0}\}$ and thus, $G_f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ induces an endomorphism $g_f : \mathbb{P}^n \to \mathbb{P}^n$. Indeed, let us consider a point $\mathbf{a} \in \mathbb{C}^{n+1}$. By definition of G_f, if $G_f(\mathbf{a}) = \mathbf{0}$, then the monic polynomial $F_{\mathbf{a}}$ vanishes at $a_0, a_1, \ldots, a_{n+1}$. The critical points of $F_{\mathbf{a}}$ are those points a_k for which $m_k > 0$. They are all mapped to 0 and thus, $F_{\mathbf{a}}$ has only one critical value in $\mathbb{C}$, namely 0.

Lemma 4.0.1. *Let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree d. Suppose that h has only two critical values, say 0 and ∞. Then h is conjugate to $z \mapsto z^d$.*

Proof of lemma 4.0.1. For each $z \in \mathbb{P}^1$, let m_z denote the local degree of h at z. By a quick local degree calculation, we have

$$\sum_{z \,\in\, h^{-1}(0)} m_z \leq d \qquad \text{and} \qquad \sum_{z \,\in\, h^{-1}(\infty)} m_z \leq d.$$

The Riemann-Hurwitz formula implies that

$$\sum_{z \,\in\, h^{-1}(0)} (m_z - 1) + \sum_{z \,\in\, h^{-1}(\infty)} (m_z - 1) = \left(\sum_{z \,\in\, h^{-1}(0)} m_z + \sum_{z \,\in\, h^{-1}(\infty)} m_z \right)$$
$$- \big(|h^{-1}(0)| + |h^{-1}(\infty)|\big) = 2d - 2.$$

Combining the above, we have

$$2d - 2 \leq 2d - \big(|h^{-1}(0)| + |h^{-1}(\infty)|\big) \implies |h^{-1}(0)| + |h^{-1}(\infty)| \leq 2$$
$$\implies |h^{-1}(0)| = |h^{-1}(\infty)| = 1.$$

So h is a rational function with two critical points, each of multiplicity $d-1$, hence $h(z) = A(z-c)^d$, or $h(z) = A/(z-c)^d$. $\square$

If $G_f(\mathbf{a}) = \mathbf{0}$, then $F_{\mathbf{a}}$ is a monic polynomial of degree d with only one critical value, 0, in $\mathbb{C}$. Hence all preimages of this critical value must coincide and since $a_0 = 0$, they all coincide at 0: $a_0 = a_1 = \ldots = a_{n+1} = 0$, so $G_f^{-1}\big(\mathbf{0}\big) = \{\mathbf{0}\}$. Therefore the map $g_f : \mathbb{P}^n \to \mathbb{P}^n$ induced by $G_f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is an endomorphism given in homogeneous coordinates as:

$$g_f : [x_1 : \ldots : x_{n+1}] \longmapsto [F_{\mathbf{x}}(x_{\nu(1)}) : \ldots : F_{\mathbf{x}}(x_{\nu(n+1)})].$$

Let us now prove that for all $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$, we have

$$\pi(\tau) = g_f \circ \pi \circ \sigma_f(\tau).$$

Let τ be a point in $\mathrm{Teich}(S^2, \mathcal{P}_f)$ and set $\tau' := \sigma_f(\tau)$.

We will show that there is a representative ϕ of τ and a representative ψ of τ' such that $\phi(\infty) = \psi(\infty) = \infty$, $\phi(p_0) = \psi(p_0) = 0$ and

$$G_f\big(\psi(p_1), \ldots, \psi(p_{n+1})\big) = \big(\phi(p_1), \ldots, \phi(p_{n+1})\big). \tag{4.1}$$

It then follows that

$$g_f\big([\psi(p_1) : \ldots : \psi(p_{n+1})]\big) = [\phi(p_1) : \ldots : \phi(p_{n+1})]$$

which concludes the proof since

$$\pi(\tau') = [\psi(p_1) : \ldots : \psi(p_{n+1})] \quad \text{and} \quad \pi(\tau) = [\phi(p_1) : \ldots : \phi(p_{n+1})].$$

To show the existence of ϕ and ψ, we may proceed as follows. Let ϕ be any representative of τ such that $\phi(\infty) = \infty$ and $\phi(p_0) = 0$. Then, there is a representative $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$ of τ' and a rational map

$$F : (\mathbb{P}^1, \psi(\mathcal{P}_f)) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$$

such that the following diagram commutes:

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f))
\end{tikzcd}$$

We may normalize ψ so that $\psi(\infty) = \infty$ and $\psi(p_0) = 0$. Then, F is a polynomial of degree d. Multiplying ψ by a nonzero complex number, we may assume that F is a monic polynomial.

We now check that these homeomorphisms ϕ and ψ satisfy the required Property (4.1). For $k \in [0, n+1]$, set

$$x_k := \psi(p_k) \quad \text{and} \quad y_k := \phi(p_k).$$

We must show that

$$G_f(x_1, \ldots, x_{n+1}) = (y_1, \ldots, y_{n+1}).$$

Note that, for $k \in [0, n+1]$, we have the following commutative diagram:

$$\begin{tikzcd} p_{\nu(k)} \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "f"'] & x_{\nu(k)} \arrow[d, mapsto, "F"] \\ p_k \arrow[r, mapsto, "\phi"] & y_k \end{tikzcd}$$

Consequently, $F(x_{\nu(k)}) = y_k$. In particular $F(x_{\nu(0)}) = 0$. In addition, the critical points of F are exactly those points x_k for which $m_k > 0$, counted with multiplicity m_k. As a consequence, $F = F_{\mathbf{x}}$ and

$$G_f \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} F_{\mathbf{x}}(x_{\nu(1)}) \\ \vdots \\ F_{\mathbf{x}}(x_{\nu(n+1)}) \end{pmatrix} = \begin{pmatrix} F(x_{\nu(1)}) \\ \vdots \\ F(x_{\nu(n+1)}) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix}.$$

In this section of the proof of theorem 4.0.1, we prove that the critical locus of g_f is contained in Δ. Recall that the *critical locus of* g_f, C_1, is the set of points in $\mathbb{P}^n$ where the Jacobian vanishes.

To see that the critical locus of g_f is contained in Δ, we must show that $\mathrm{Jac}\ G_f : \mathbb{C}^{n+1} \to \mathbb{C}$ does not vanish outside Δ.

Note that since $G_f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is homogeneous, $\mathrm{Jac}\ G_f(x_1, \ldots, x_{n+1})$ is a homogeneous polynomial of degree $(n+1) \cdot (d-1)$ in the variables $x_1, \ldots, x_{n+1}$. Consider the polynomial $J \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ defined by

$$J(x_1, \ldots, x_{n+1}) := \prod_{0 \le i < j \le n+1} (x_i - x_j)^{m_i + m_j} \quad \text{with} \quad x_0 := 0.$$

Proposition 4.0.1. *The Jacobian* $\mathrm{Jac}\ G_f$ *is divisible by* J.

Proof of proposition 4.0.1. Set $x_0 := 0$ and $G_0 := 0$. For $j \in [1, n+1]$, let G_j be the j-th coordinate of $G_f(x_1, \dots, x_{n+1})$, that is

$$G_j := d \int_{x_{\nu(0)}}^{x_{\nu(j)}} \prod_{k=0}^{n+1} (w - x_k)^{m_k} \mathrm{d}w.$$

For $0 \leq i < j \leq n+1$, note that setting $w = x_i + t(x_j - x_i)$, we have

$$\begin{aligned} G_{\mu(j)} - G_{\mu(i)} &= d \int_{x_i}^{x_j} \prod_{k=0}^{n+1} (w - x_k)^{m_k} \mathrm{d}w \\ &= d \int_0^1 \prod_{k=0}^{n+1} (x_i + t(x_j - x_i) - x_k)^{m_k} \cdot (x_j - x_i) \mathrm{d}t \\ &= (x_j - x_i)^{m_i + m_j + 1} \cdot H_{i,j} \end{aligned}$$

with

$$H_{i,j} := d \int_0^1 t^{m_i} (t-1)^{m_j} \prod_{\substack{k \in [0, n+1] \\ k \neq i, j}} \big(x_i - x_k + t(x_j - x_i)\big)^{m_k} \mathrm{d}t.$$

In particular, $G_{\mu(j)} - G_{\mu(i)}$ is divisible by $(x_j - x_i)^{m_i + m_j + 1}$.

For $k \in [0, n+1]$, let L_k be the row defined as:

$$L_k := \begin{bmatrix} \dfrac{\partial G_k}{\partial x_1} & \dots & \dfrac{\partial G_k}{\partial x_{n+1}} \end{bmatrix}.$$

Note that L_0 is the zero row, and for $k \in [1, n+1]$, L_k is the k-th row of the Jacobian matrix of G_f. According to calculations above, the entries of $L_{\mu(j)} - L_{\mu(i)}$ are the partial derivatives of $(x_j - x_i)^{m_i + m_j + 1} \cdot H_{i,j}$. It follows that $L_{\mu(j)} - L_{\mu(i)}$ is divisible by $(x_j - x_i)^{m_i + m_j}$. Indeed, $L_{\mu(j)} - L_{\mu(i)}$ is either the difference of two rows of the Jacobian matrix of G_f, or such a row up to sign, when $\mu(i) = 0$ or $\mu(j) = 0$. As a consequence, Jac G_f is divisible by J. $\square$

Since $\sum m_j = d-1$, the lemma below shows that the degree of J is $(n+1) \cdot (d-1)$.

Lemma 4.0.2. *The degree of J is $(n+1) \cdot (d-1)$.*

Proof of lemma 4.0.2. The proof is just a simple calculation.

$$\begin{aligned}
\sum_{0\le i<j<\le n+1}(m_i+m_j) &= \sum_{j=0}^{n+1}\sum_{i=0}^{j-1} m_j + \sum_{i=0}^{n+1}\sum_{j=i+1}^{n+1} m_i \\
&= \sum_{j=0}^{n+1} j m_j + \sum_{i=0}^{n+1}(n+1-i)m_i \\
&= \sum_{k=0}^{n+1} k m_k + \sum_{k=0}^{n+1}(n+1-k)m_k \\
&= (n+1)\sum_{k=0}^{n+1} m_k = (n+1)\cdot(d-1).
\end{aligned}$$

□

Since J and Jac G_f are homogeneous polynomials of the same degree and since J divides Jac G_f, they are equal up to multiplication by a nonzero complex number. This shows that Jac G_f vanishes exactly when J vanishes.

Corollary 4.0.1. *Recall that*

$$m_i := \text{loc deg } f|_{p_i} - 1.$$

Set $x_0 := 0$. The critical locus of g_f is precisely

$$C_1 = \{[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n \mid x_i = x_j, \text{ and } m_i + m_j > 0 \text{ for } 0 \le i < j \le n+1\},$$

and we therefore have $C_1 \subseteq \Delta$.

This corollary follows immediately from the computations above.

To see that $g_f(\Delta) \subseteq \Delta$, let $\mathbf{x} := (x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1}$ and set $x_0 := 0$. Set $(y_0, y_1, \ldots, y_{n+1}) := \big(0, F_{\mathbf{x}}(x_{\nu(1)}), \ldots, F_{\mathbf{x}}(x_{\nu(n+1)})\big)$. Then,

$$G_f(x_1, \ldots, x_{n+1}) = (y_1, \ldots, y_{n+1}).$$

Note that

$$x_i = x_j \quad \Longrightarrow \quad y_{\mu(i)} = y_{\mu(j)}.$$

This follows immediately from the formula of the map G_f. In addition, the point $[x_1 : \ldots : x_{n+1}]$ belongs to Δ precisely when there are integers $i \neq j$ in $[0, n+1]$ such that $x_i = x_j$. As a consequence,

$$[x_1 : \ldots : x_{n+1}] \in \Delta \quad \Longrightarrow \quad [y_1 : \ldots : y_{n+1}] \in \Delta.$$

This proves that $g_f(\Delta) \subseteq \Delta$. The proof of theorem 4.0.1 is complete. $\square$

4.1 Periodic components

For the maps $g_f : \mathbb{P}^n \to \mathbb{P}^n$ defined above, every component of the forbidden locus is periodic. We state this in the following proposition.

Proposition 4.1.1. *Let $g_f : \mathbb{P}^n \to \mathbb{P}^n$ be a postcritically finite endomorphism constructed in theorem 4.0.1. Then all components of Δ are*

1. *periodic, and*
2. *each such periodic cycle contains a critical component of g_f.*

Proof. We begin with the proof of the first point above. Define $x_0 := 0$. Recall that $\Delta := \{[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n \ : \ \exists\ i, j \in [0, n+1], i \neq j \text{ with } x_i = x_j\}$. As mentioned above, we see that if $x_i = x_j$ for some $i, j \in [0, n+1]$ where $i \neq j$, then

$$y_{\mu(i)} = y_{\mu(j)},$$

where y is the 'range' coordinate. Since we wish to iterate the map g_f, we identify the domain and the range, and we reformulate the previous remark as

$$g_f : x_i = x_j \mapsto x_{\mu(i)} = x_{\mu(j)}.$$

Since R_f is a periodic portrait, every postcritical point of f is contained in a periodic cycle. Define N_k to be the length of the periodic cycle containing the postcritical point p_k:

$$N_k := \min\{l \geq 0 \ : \ f^{\circ l}(p_k) = p_k\}.$$

Then the hyperplane $x_i = x_j \in \Delta$ is periodic of period $N := \mathrm{lcm}(N_i, N_j)$ since for any $r > 0$,

$$g_f^{\circ r} : x_i = x_j \mapsto x_{\mu^{\circ r}(i)} = x_{\mu^{\circ r}(j)} \implies g_f^{\circ N} : x_i = x_j \mapsto x_{\mu^{\circ N}(i)} = x_{\mu^{\circ N}(j)}.$$

The point p_i is periodic of period N_i under f and p_j is periodic of period N_j under f, so $x_{\mu^{\circ N}(i)} = x_i$ and $x_{\mu^{\circ N}(j)} = x_j$. Moreover, N is the minimal such number, so the component $x_i = x_j \in \Delta$ is periodic of period N.

We now prove that each periodic cycle of hyperplanes in Δ, contains a critical component of g_f. Recall that the critical locus of g_f is

$$C_1 = \{[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n \mid x_i = x_j, \text{ and } m_i + m_j > 0 \text{ for } 0 \leq i < j \leq n+1\},$$

where m_i is the multiplicity of p_i if $p_i \in \Omega_f$, and $m_i := 0$ if $p_i \notin \Omega_f$.

Let $x_i = x_j \in \Delta$. Since R_f is periodic, p_i is contained in a periodic cycle of period N_i. Since $p_i \in \mathcal{P}_f$, there is $M \geq 0$ such that

$$p_{\mu^{\circ M}(i)} = f^{\circ M}(p_i)$$

is critical. Then by corollary 4.0.1, the hyperplane

$$x_{\mu^{\circ M}(i)} = x_{\mu^{\circ M}(j)}$$

is critical, and so we see that $g_f^{\circ M}$ maps the original hyperplane $x_i = x_j$ to the critical hyperplane $x_{\mu^{\circ M}(i)} = x_{\mu^{\circ M}(j)}$. So the periodic cycle containing $x_i = x_j$ also contains a critical component of g_f. □

We have the following as an immediate corollary.

Corollary 4.1.1. *Let $g_f : \mathbb{P}^n \to \mathbb{P}^n$ be a postcritically finite endomorphism constructed in theorem 4.0.1. Then the postcritical locus of g_f is equal to the forbidden locus, Δ.*

Proof. This is a direct consequence of the proposition above. □

This corollary has important implications which we discuss in the following section.

4.2 Kobayashi hyperbolicity

For complex dynamics in one variable, one very useful fact is that if a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ is postcritically finite, and $|P_F| > 3$, then the Poincaré metric on $\mathbb{P}^1 - P_F$ is expanded by F. In [3], C. McMullen asked about constructing analogous examples in $\mathbb{P}^n$: construct $F : \mathbb{P}^n \to \mathbb{P}^n$ such that the complement of the postcritical locus is Kobayashi hyperbolic. In [13], Fornæss and Sibony thoroughly analyze the dynamics of two examples. In [23], we proved that the endomorphisms $g_f : \mathbb{P}^n \to \mathbb{P}^n$ constructed in the unicritical case all have this property. Each of the endomorphisms we constructed in theorem 4.0.1 also has this property.

Definition 4.2.1. *Given any complex manifold M, the* Kobayashi metric *on M is the largest metric such that every holomorphic map $h : \mathbb{D} \to M$ satisfies*

$$||h'(0)|| \leq 1.$$

The manifold M is Kobayashi hyperbolic *if this metric is nowhere degenerate.*

Corollary 4.2.1. *Let $g_f : \mathbb{P}^n \to \mathbb{P}^n$ be a postcritically finite endomorphism consructed in theorem 4.0.1. Then the complement of the postcritical locus of g_f is $\mathbb{P}^n - \Delta$, which is Kobayashi hyperbolic.*

Proof. The result follows directly from the Teichmüller theory and Royden's theorem: the complement of the postcritical locus of g_f is the moduli space, $\mathrm{Mod}(S^2, \mathcal{P}_f)$ and is therefore Kobayashi hyperbolic. □

In [15], Green proves that the complement of $2n + 1$ hyperplanes in $\mathbb{P}^n$ is Kobayashi hyperbolic if the hyperplanes are in general position. From proposition 3.2.1, we see that if $|\mathcal{P}_f| > 4$, then

$$\frac{(|\mathcal{P}_f| - 1)(|\mathcal{P}_f| - 2)}{2} \geq 2(|\mathcal{P}_f| - 3) + 1$$

where the quantity on the left is the number of hyperplanes contained in Δ and the quantity on the right is that from Green's theorem. If the hyperplanes of Δ were in general position, then Green's theorem would imply the above result. However, the hyperplanes of Δ are not in general position, so this result does not apply, and we use the Teichmüller theory to obtain the result.

Remark 4.2.1. We have provided infinitely many examples, of nontrivial endomorphisms of $\mathbb{P}^n$ such that the complement of the postcritical locus is Kobayashi hyperbolic. Moreover, the methods presented in this thesis can be used to recover both of the Fornæss and Sibony examples. There is also a family of postcritically finite endomorphisms found by S. Crass in [8]. This family can be recovered by the methods in theorem 4.0.1 as well.

CHAPTER 5

THE $\pi\sigma$-PROPERTY

The calculation in theorem 4.0.1 inspires us to define the following property which was the essence of the construction in the proof of the theorem. This will be very significant for the work that follows. Let f be a Thurston map of topological degree d, with postcritical set $\mathcal{P}_f = \{p_1, \dots, p_{n+3}\}$. We identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way (see section 3.2.2).

Let $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$, and let $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ be a representative homeomorphism of τ, which is normalized in the standard way. Then there exists a unique normalized $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$ such that the following diagram commutes, and $F_\phi : (\mathbb{P}^1, \psi(\mathcal{P}_f)) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a rational function of degree d.

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F_\phi"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f))
\end{tikzcd}$$

Moreover, ψ represents $\tau' := \sigma_f(\tau)$.

Note that F_ϕ is naturally a holomorphic function of $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$, however, it can sometimes be expressed as a holomorphic function of just $\pi\left(\sigma_f(\tau)\right)$, and this is the essential observation. Define $x_i := \psi(p_i)$, and $y_i := \phi(p_i)$ for $i \in [1, n+1]$; where we naturally consider $[x_1 : \dots : x_n : 1]$ to be in the subset of $\mathbb{P}^n - \Delta$ corresponding to $\pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f)))$, and $[y_1 : \dots ; y_n : 1]$ to be in $\mathbb{P}^n - \Delta$.

Definition 5.0.2. *We say f has* the $\pi\sigma$-property *if the rational function F_ϕ depends only on $\mathbf{x} := \pi(\sigma_f(\tau))$, for $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$.*

Note that if F_ϕ depends only on $[x_1 : \dots : x_n : 1]$, then this dependence is naturally holomorphic for $[x_1 : \dots : x_n : 1] \in \pi\left(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))\right)$

since F_ϕ varies holomorphically with $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$, and both of the maps $\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f)$ and $\pi : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathbb{P}^n - \Delta$ are holomorphic.

One may naturally wonder if the definition above depends on the normalization.

Remark 5.0.2. Suppose that f has the $\pi\sigma$-property; then F_ϕ depends on $\psi(\mathcal{P}_f)$. Suppose we normalize differently. Choose a Möbius transformation $\mu_{i,j,k}$ so that for $\mu_{i,j,k}(\psi(p_i)) = 0$, $\mu_{i,j,k}(\psi(p_j)) = 1$ and $\mu_{i,j,k}(\psi(p_k)) = \infty$. Then clearly, the map $F_{\mu\circ\phi}$ now depends on $\mu_{i,j,k}(\psi(\mathcal{P}_f))$.

In the proof of theorem 4.0.1, we saw that if f is a topological polynomial with $\Omega_f \subseteq \mathcal{P}_f$, then f has the $\pi\sigma$-property, and in this particular case, we exploited the fact that F_ϕ induces a map $g_f : \mathbb{P}^n \to \mathbb{P}^n$. We now further explore the idea of an induced map.

5.1 The induced map

For each

$$\mathbf{x} = [x_1 : \ldots : x_n : 1] \in \pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))),$$

$$F_\phi : (\mathbb{P}^1, \{0, 1, \infty, x_1, \ldots, x_n\}) \longrightarrow (\mathbb{P}^1, \{0, 1, \infty, y_1, \ldots, y_n\}),$$

where the y_i are as above. Consider Ω_{F_ϕ}, which is the set of critical points of the map F_ϕ. Then

$$F_\phi\left(\Omega_{F_\phi} \cup \{0, 1, \infty, x_1, \ldots, x_n\}\right) = \{0, 1, \infty, y_1, \ldots, y_n\}.$$

If f has the $\pi\sigma$-property, sometimes the rational function F_ϕ actually induces a map

$$g_f : \pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))) \longrightarrow \pi(\mathrm{Teich}(S^2, \mathcal{P}_f)),$$

given by

$$g_f : [x_1 : \ldots : x_n : 1] \longmapsto [y_1 : \ldots : y_n : 1].$$

This map is obtained by evaluation of the rational map F_ϕ at the elements of

$$\Omega_{F_\phi} \cup \{0, 1, \infty, x_1, \ldots, x_n\},$$

according to the portrait of f, and the commutative diagram.

Definition 5.1.1. *Let $f : S^2 \to S^2$ be a Thurston map of topological degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, such that there is a map*

$$g_f : \pi\left(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))\right) \longrightarrow \mathbb{P}^n - \Delta$$

given by

$$g_f : [x_1 : \ldots : x_n : 1] \longmapsto [y_1 : \ldots : y_n : 1].$$

Then we say that F_ϕ induces a map *if $g_f : \pi\left(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))\right) \longrightarrow \mathbb{P}^n - \Delta$ extends to a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$.*

Remark 5.1.1. It follows from remark 5.0.2 that the existence of an induced map is independent of normalization.

5.1.1 Homogeneous coordinates & topological polynomials

Since we are working with $\mathbb{P}^n$, we will proceed to reformulate the above for topological polynomials, in terms of homogeneous coordinates. Let the Thurston map

$f : S^2 \to S^2$ is be a topological polynomial of degree d, with postcritical set $\mathcal{P}_f$. Identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way (see section 3.2.2).

Let $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$, and let $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ be a representative homeomorphism of τ, normalized in the standard way. There is a unique homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$, normalized in the standard way such that the following diagram commutes, where $F_\phi : (\mathbb{P}^1, \psi(\mathcal{P}_f)) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a polynomial of degree d,

$$\begin{tikzcd} (S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F_\phi"] \\ (S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f)) \end{tikzcd}$$

and ψ represents $\tau' := \sigma_f(\tau)$.

Define $x_i := \psi(p_i)$, and $y_i := \phi(p_i)$ for $i \in [1, n+1]$; where we naturally consider $\mathbf{x} := [x_1 : \ldots : x_n : 1]$ to be in the subset of $\mathbb{P}^n - \Delta$ corresponding to $\pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f)))$, and $\mathbf{y} := [y_1 : \ldots : y_n : 1]$ to be in $\mathbb{P}^n - \Delta$. The map f has the $\pi\sigma$-property if the coefficients of F_ϕ depend only on $[x_1 : \ldots : x_n : 1] \in \pi(\sigma_f(\tau))$, for $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$. We write

$$F_\phi(z) = \alpha_d(\mathbf{x})z^d + \ldots + \alpha_i(\mathbf{x})z^i + \ldots + \alpha_0(\mathbf{x})$$

where $\alpha_i : \pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))) \longrightarrow \mathbb{C}$ is holomorphic; in fact, for each $i \in [0, d]$, α_i is a rational function of the x_i. If there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, which extends the map

$$g_f : \pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))) \longrightarrow \pi(\mathrm{Teich}(S^2, \mathcal{P}_f)),$$

then there are rational functions $A_i : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}$, which are homogeneous such that if $\mathbf{w} \in \mathbb{C}^{n+1}$ is a representative of $\mathbf{x} \in \pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f)))$, then $A_i(\mathbf{w}) = \alpha_i(\mathbf{x})$.

The functions A_i are the homogeneous versions of the α_i, that is,

$$A_i(w_1, \ldots, w_{n+1}) := \alpha_i([w_1/w_{n+1} : \ldots : w_n/w_{n+1} : 1]).$$

Each A_i is a ratio of two homogeneous polynomials, p_i and q_i, which we may assume have no common factors $p_i, q_i : \mathbb{C}^{n+1} \to \mathbb{C}$, where $\deg(p_i) = \deg(q_i)$, so $\deg(A_i) := \deg(p_i) - \deg(q_i) = 0$. Consider the polynomial

$$F_{\mathbf{w}}(z) = A_d(\mathbf{w})z^d + \ldots + A_i(\mathbf{w})(w_{n+1})^{d-i}z^i + \ldots + A_0(\mathbf{w})(w_{n+1})^d.$$

This polynomial is called the *homogeneous polynomial associated to* F_ϕ. Observe that this polynomial is homogeneous of degree d in the variables z and w_i, for $i \in [1, n+1]$. We see by construction that if the topological polynomial f has the $\pi\sigma$-property, then there exists a unique such polynomial $F_{\mathbf{w}}$.

If there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, then $F_{\mathbf{w}}$ induces a homogeneous map

$$G_f : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$$

by evaluation. Let $\mathbf{w} \in \mathbb{C}^{n+1}$ for which the polynomial $F_{\mathbf{w}}$ exists ($F_{\mathbf{w}}$ is not defined for all $\mathbf{w} \in \mathbb{C}^{n+1}$ as the A_i may have denominators), and let $\Omega_{F_{\mathbf{w}}}$ be the set of critical points of the polynomial $F_{\mathbf{w}}$. Then

$$F_{\mathbf{w}}\left(\Omega_{F_{\mathbf{w}}} \cup \{0, \infty, w_1, \ldots, w_{n+1}\}\right) = \{v_1, \ldots, v_{n+1}\}$$

induces the map G_f, that is, $G_f(\mathbf{w}) = \mathbf{v}$, and the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}^{n+1} - \{\mathbf{0}\} \arrow[r, dashed, "G_f"] \arrow[d] & \mathbb{C}^{n+1} \arrow[d, dashed] \\
\mathbb{P}^n \arrow[r, dashed, "g_f"] & \mathbb{P}^n
\end{tikzcd}$$

The map G_f, which is induced by the homogeneous polynomial, will be called the *map induced by* $F_{\mathbf{w}}$.

Definition 5.1.2. *Let the Thurston map f be a topological polynomial of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with* $\mathbb{P}^n - \Delta$ *in the standard way. Suppose that f has the $\pi\sigma$-property. The* homogeneous polynomial associated to F_ϕ *is the particular polynomial $F_{\mathbf{w}}$ constructed above. If there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, then the $F_{\mathbf{w}}$ induces a map $G_f : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$, called the* map induced by $F_{\mathbf{w}}$.

If the topological polynomial f has the $\pi\sigma$-property, we define another polynomial we will require in subsequent discussions.

Definition 5.1.3. *Let the Thurston map f be a topological polynomial of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with* $\mathbb{P}^n - \Delta$ *in the standard way. Suppose that f has the $\pi\sigma$-property. The* monic polynomial associated to F_ϕ *is the monic polynomial*

$$\widetilde{F_{\mathbf{w}}} := \frac{F_{\mathbf{w}}}{A_d(\mathbf{w})}.$$

We may express $\widetilde{F_{\mathbf{w}}}$ as

$$\widetilde{F_{\mathbf{w}}}(z) = z^d + B_{d-1}(\mathbf{w})z^{d-1} + \ldots + B_i(\mathbf{w})z^i + \ldots + B_0(\mathbf{w})$$

where $B_i : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}$ is a rational function; in particular,

$$B_i(\mathbf{w}) := \frac{A_i(\mathbf{w})(w_{n+1})^{d-i}}{A_d(\mathbf{w})}.$$

Notice that each B_i is homogeneous of degree $d - i$; that is $B_i(\mathbf{w}) = s_i(\mathbf{w})/t_i(\mathbf{w})$, where we may assume that s_i and t_i have no common factors, $s_i, t_i : \mathbb{C}^{n+1} \to \mathbb{C}$ are homogeneous polynomials such that $\deg(s_i) = d - i + \deg(t_i)$ for all $i \in [0, d-1]$.

We will frequently use the polynomial $\widetilde{F_{\mathbf{w}}}$. Notice that if there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, then $\widetilde{F_{\mathbf{w}}}$ induces a homogeneous map

$$\widetilde{G_f} : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$$

by evaluation as

$$\widetilde{G_f}(\mathbf{w}) := \frac{1}{A_d(\mathbf{w})} G_f(\mathbf{w})$$

and G_f was induced by evaluation as explained above. Hence, we also have this commutative diagram.

$$\begin{tikzcd}
\mathbb{C}^{n+1} - \{\mathbf{0}\} \arrow[r, dashed, "\widetilde{G_f}"] \arrow[d] & \mathbb{C}^{n+1} \arrow[d, dashed] \\
\mathbb{P}^n \arrow[r, dashed, "g_f"] & \mathbb{P}^n
\end{tikzcd}$$

Proposition 5.1.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Then g_f is unique.*

Proof. This follows from the uniqueness of the map F_ϕ, and the uniqueness of the homogeneous polynomial $F_{\mathbf{w}}$ which induces the map $G_f : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$, which completely determines the map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$.

$\square$

5.1.2 The algebraic degree of $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$

Recall that if $g : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a rational map, then there is a map $G : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, whose coordinate functions $G_i(w_1, \ldots, w_{n+1})$ are homogeneous polynomials with no common factor, (that is, there is no polynomial $p(w_1, \ldots, w_{n+1})$ which divides all of the $G_i(w_1, \ldots, w_{n+1})$), such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}^{n+1} - \{\mathbf{0}\} \arrow[r, "G"] \arrow[d] & \mathbb{C}^{n+1} \arrow[d, dashed] \\ \mathbb{P}^n \arrow[r, dashed, "g"] & \mathbb{P}^n \end{tikzcd}$$

The map G is unique up to scaling by a nonzero complex number.

Definition 5.1.4. *The degree of the homogeneous polynomial G_i is equal to the* algebraic degree of the map g.

That is, the algebraic degree of g is equal to the degree of the homogeneous map G which gives g in homogeneous coordinates.

Let the Thurston map f be a topological polynomial of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, such that there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Let $\widetilde{F_{\mathbf{w}}}$ be the monic polynomial associated to F_ϕ. We express the monic polynomial as

$$\widetilde{F_{\mathbf{w}}}(z) = z^d + B_{d-1}(\mathbf{w})z^{d-1} + \ldots + B_i(\mathbf{w})z^i + \ldots + B_0(\mathbf{w})$$

where $B_i(\mathbf{w}) = s_i(\mathbf{w})/t_i(\mathbf{w})$, s_i and t_i have no common factors, $s_i, t_i : \mathbb{C}^{n+1} \to \mathbb{C}$ are homogeneous polynomials such that $\deg(s_i) = d - i + \deg(t_i)$ for all $i \in [0, d-1]$.

Define $b_d(\mathbf{w}) := \mathrm{lcm}\{s_i(\mathbf{w})\}_{i=0}^{d-1}$, which is homogeneous, and consider the polynomial

$$s_d(\mathbf{w}) \cdot \widetilde{F_{\mathbf{w}}}(z) = b_d(\mathbf{w})z^d + \ldots + b_i(\mathbf{w})z^i + \ldots + b_0(\mathbf{w}),$$

where each $b_i(\mathbf{w})$ is a homogeneous polynomial. We can immediately see that

$$\text{alg deg } (g_f) \geq d + \deg(s_d(\mathbf{w})),$$

so in general, the algebraic degree of g_f is at least equal to the topological degree of f.

Proposition 5.1.2. *Let the Thurston map f be a topological polynomial of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with* $\mathbb{P}^n - \Delta$ *in the standard way. Suppose that f has the $\pi\sigma$-property, such that there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Let $\widetilde{F_{\mathbf{w}}}$ be the monic polynomial associated to F_ϕ. We express the monic polynomial as*

$$\widetilde{F_{\mathbf{w}}}(z) = z^d + B_{d-1}(\mathbf{w})z^{d-1} + \ldots + B_i(\mathbf{w})z^i + \ldots + B_0(\mathbf{w})$$

The map $\widetilde{G_f} : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$ induced by the monic polynomial is holomorphic if and only if

$$\text{alg deg } (g_f) = d.$$

Proof. The proof follows immediately from the discussion above. As previously mentioned, alg deg (g_f) is at least d. Recall also that the degree of the induced map $\widetilde{G_f}$ is equal to d. So if $\widetilde{G_f} : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$ is holomorphic, then we must have alg deg $(g_f) = d$.

Conversely, suppose that $\widetilde{G_f} : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$ is not holomorphic. Then there is a homogeneous polynomial of minimal degree, $p(\mathbf{w})$ (defined up to scaling by a nonzero complex number), such that the map H defined by $H_i := p(\mathbf{w}) \cdot \widetilde{G_i}(\mathbf{w})$, is holomorphic. Then clearly, we have

$$\text{alg deg } (g_f) = d + \deg(p).$$

$\square$

We will return to this discussion in subsequent sections.

Corollary 5.1.1. *Let the Thurston map f be a topological polynomial of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with* $\mathbb{P}^n - \Delta$ *in the standard way. Suppose that f has* *the $\pi\sigma$-property, such that there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Let $\widetilde{F_{\mathbf{w}}}$ be the monic polynomial associated to F_ϕ. We express it as*

$$\widetilde{F_{\mathbf{w}}}(z) = z^d + \frac{p_{d-1}(\mathbf{w})}{q_{d-1}(\mathbf{w})} z^{d-1} + \ldots + \frac{p_0(\mathbf{w})}{q_0(\mathbf{w})}.$$

where p_i, q_i are homogeneous. Suppose that alg deg $(g_f) = d$. *Then* $\deg(q_i) = 0$ *for all $i \in [1, d-1]$.*

Proof. This is also clear from the discussion above. $\square$

We now present an example.

Example 5.1.1. Let $f : S^2 \to S^2$ be a Thurston map with the postcritical set $\mathcal{P}_f = \{0, 1, \infty, p\}$, which realizes the following ramification portrait.

$$\begin{tikzcd} 0 \arrow[r, "2"] & 1 \arrow[r] & p \arrow[ll, bend left] \end{tikzcd} \qquad\qquad \begin{tikzcd} \infty \arrow[loop right, "2"] \end{tikzcd}$$

This is the ramification portrait of the *rabbit*. Let $\tau \in \text{Teich}(S^2, \mathcal{P}_f)$, and let $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ be a representative homeomorphism, normalized so that $\phi(0) = 0, \phi(1) = 1$, and $\phi(\infty) = \infty$. Then there is a unique homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$, so that $\psi(0) = 0, \psi(1) = 1$, and $\psi(\infty) = \infty$, such that the following diagram commutes, where F_ϕ is a quadratic polynomial,

$$\begin{tikzcd} (S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F_\phi"] \\ (S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f)) \end{tikzcd}$$

and ψ represents $\tau' := \sigma_f(\tau)$.

Define $x_1 := \psi(p)$; we identify $\text{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^1 - \{0, 1, \infty\}$. We conclude that F_ϕ must have the following form:

$$F_\phi(z) = Az^2 + 1,$$

where A is a complex parameter which depends on f and ϕ. Observe from the commutative diagram above that $F_\phi(x_1) = 0$, so

$$A = -\frac{1}{x_1^2},$$

and thus

$$F_\phi(z) = -\frac{z^2}{x_1^2} + 1.$$

We immediately see that f has the $\pi\sigma$-property. We now find the homogeneous polynomial $F_{\mathbf{w}}$ and the monic polynomial $\widetilde{F_{\mathbf{w}}}$, associated to F_ϕ. A quick calculation reveals that

$$F_{\mathbf{w}}(z) = -\frac{w_2^2 z^2}{w_1^2} + w_2^2.$$

Notice that this polynomial is homogeneous in w_1, w_2 and z; it is homogeneous of degree 2. The monic polynomial is

$$\widetilde{F_{\mathbf{w}}}(z) = z^2 - w_1^2.$$

Observe that $\Omega_{F_\phi} = \{0, \infty\}$. The polynomial $\widetilde{F_{\mathbf{w}}}$ maps the set $\{0, \infty, w_1, w_2\}$ to the set $\{0, \infty, v_1, v_2\}$, which induces the map $\widetilde{G_f} : \mathbb{C}^2 \to \mathbb{C}^2$:

$$\widetilde{G_f}(w_1, w_2) = (v_1, v_2) \text{ where } v_1 := \widetilde{F_{\mathbf{w}}}(w_2) \text{ and } v_2 := \widetilde{F_{\mathbf{w}}}(0),$$

that is

$$\widetilde{G_f} : (w_1, w_2) \longmapsto (w_2^2 - w_1^2, -w_1^2),$$

and we can see that $\widetilde{G_f}$ is holomorphic, and induces the map

$$g_f : \mathbb{P}^1 \to \mathbb{P}^1, \qquad g_f[x_1 : x_2] \mapsto [x_2^2 - x_1^2 : -x_1^2].$$

Notice that the algebraic degree of g_f is equal to 2.

For an example of a Thurston map f of degree d which has the $\pi\sigma$-property such that the algebraic degree of the induced map is not equal to d, please see example 9.1.2.

5.1.3 The graph of $\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f)$

One immediate consequence of definition 5.1.1 is the following proposition.

Proposition 5.1.3. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose f is has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ as outlined above. Then the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, dashed, "g_f"]
\end{tikzcd}$$

Proof. This follows immediately from the definition of the induced map. $\square$

We now see that the graph of σ_f in $\mathrm{Teich}(S^2, \mathcal{P}_f) \times \mathrm{Teich}(S^2, \mathcal{P}_f)$ covers an algebraic subvariety of $\mathrm{Mod}(S^2, \mathcal{P}_f) \times \mathrm{Mod}(S^2, \mathcal{P}_f)$. We paraphrase a proposition proved by C. McMullen in [29].

Proposition 5.1.4. *Let $f : S^2 \to S^2$ be a Thurston map of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Consider the map*

$$\Pi : \mathrm{Teich}(S^2, \mathcal{P}_f) \times \sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f)) \longrightarrow \mathrm{Mod}(S^2, \mathcal{P}_f) \times \mathrm{Mod}(S^2, \mathcal{P}_f)$$

given by

$$\Pi : (\tau, \sigma_f(\tau)) \longmapsto (\pi(\tau), \pi(\sigma_f(\tau))).$$

Define V_f to be the image of Π, that is,

$$V_f := \Pi\left(\mathrm{Teich}(S^2, \mathcal{P}_f) \times \sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))\right).$$

Then V_f is an irreducible algebraic subvariety of $\mathbb{P}^n \times \mathbb{P}^n$.

Proposition 5.1.5. *Let $f : S^2 \to S^2$ be a Thurston map of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with* $\mathbb{P}^n - \Delta$ *in the standard way. Suppose that f has the $\pi\sigma$-property. Let V_f be as in proposition 5.1.4, and let $\rho_2 : V_f \to \mathbb{P}^n$ be the projection onto the second factor, that is, for $(\mathbf{v}_1, \mathbf{v}_2) \in V_f$, $\rho_2(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_2$. The degree of ρ_2 is equal to 1, if and only if there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$.*

Proof. The proof is immediate from the definitions. $\square$

The above proposition actually provides an alternative definition of the induced map.

5.2 Necessary and sufficient conditions for an induced map

It is certainly necessary for f to have the $\pi\sigma$-property if F_ϕ is to induce such a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. It is natural to wonder if it is sufficient. The following example provides a negative answer.

Example 5.2.1. Let f be a Thurston map with the following ramification portrait,

$$\begin{tikzcd}
w_1 \arrow[r, "2"] & p \arrow[r] & 1 \arrow[dr] & & & \\
& & & 0 \arrow[loop right] & & \infty \arrow[loop right, "3"] \\
& w_2 \arrow[r, "2"] & q \arrow[ur] & & &
\end{tikzcd}$$

with critical set $\Omega_f = \{w_1, w_2, \infty\}$, and postcritical set $P_f = \{0, 1, p, q, \infty\}$. Let $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$ of which $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a representative homeomorphism such that $\phi(0) = 0$, $\phi(\infty) = \infty$, $\phi(1) = 1$, and define $X := \phi(p)$, $Y := \phi(q)$. Suppose that $\tau' := \sigma_f(\tau)$ of which $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$ is a representative homeomorphism such that $\psi(0) = 0$, $\psi(\infty) = \infty$, $\psi(1) = 1$, and define $x := \psi(p)$, $y := \psi(q)$, $\omega_1 := \psi(w_1)$ and $\omega_2 := \psi(w_2)$. Then according to the following commutative diagram,

$$\begin{array}{ccc} (S^2, \mathcal{P}_f) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F_\phi} \\ (S^2, \mathcal{P}_f) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(\mathcal{P}_f)) \end{array}$$

F_ϕ is a cubic polynomial. Moreover, we have the following observations:

- $F_\phi(1) = 0$
- $F_\phi(x) = 0$
- $F_\phi(0) = 0$
- $F_\phi(y) = 1$
- $F_\phi(\omega_1) = X$
- $F_\phi(\omega_2) = Y$
- the critical points of F_ϕ are ω_1 and ω_2

The first three points above imply that

$$F_\phi(z) = Az(z-1)(z-x)$$

where A is a complex parameter. However, consider the equation $F_\phi(y) = 1$; this implies that

$$A = \frac{1}{y(y-1)(y-x)},$$

so we see that the Thurston map f does have the $\pi\sigma$-property as

$$F_\phi(z) = \frac{z(z-1)(z-x)}{y(y-1)(y-x)}.$$

However, there is no induced map $g_f : \mathbb{P}^1 \to \mathbb{P}^1$ in this case. If there were an induced map, then we would be able to express X and Y in terms of x and y. We see that the critical points ω_1 and ω_2 satisfy

$$F'_\phi(z) = 3z^2 - 2(x+1)z + x = 0.$$

We strive to eliminate ω_1 and ω_2 from the equations

$$F'_\phi(\omega_1) = 0, F'_\phi(\omega_2) = 0, F_\phi(\omega_1) = X, F_\phi(\omega_2) = Y$$

in an attempt to obtain an induced map. However, one does not obtain a map in this case, but a *correspondence*

$$([x : y : 1], [X : Y : 1]) \subset \mathbb{P}^2 \times \mathbb{P}^2$$

defined by the equations:

$$-27X^2y^4x^2 - 4x^3Xy^3 + 4x^4Xy^2 - 4x^4Xy + 6Xy^3x - 27X^2y^2x^2 + 54X^2y^3x^2 - 2Xy^2x$$

$$+6Xy^3x^2 - 12Xy^2x^2 + 6Xyx^2 - 108X^2y^4x + 6Xyx^3 + 54X^2y^3x - 4Xyx - 2Xy^2x^3 +$$

$$54X^2y^5x - 4Xy^3 + 4Xy^2 + x^2 - 2x^3 - 27X^2y^6 + 54X^2y^5 - 27X^2y^4 + x^4 = 0$$

and

$$-27Y^2y^4x^2 - 4x^3Yy^3 + 4x^4Yy^2 - 4x^4Yy + 6Yy^3x - 27Y^2y^2x^2 + 54Y^2y^3x^2 - 2Yy^2x$$

$$+6Yy^3x^2 - 12Yy^2x^2 + 6Yyx^2 - 108Y^2y^4x + 6Yyx^3 + 54Y^2y^3x - 4Yyx - 2Yy^2x^3 +$$

$$54Y^2y^5x - 4Yy^3 + 4Yy^2 + x^2 - 2x^3 - 27Y^2y^6 + 54Y^2y^5 - 27Y^2y^4 + x^4 = 0.$$

Notice the symmetry between X and Y in the equations above. This is due to the fact that $X = \psi(p)$ and $Y = \psi(q)$, and p and q have completely symmetric roles in the ramification portrait. In other words, X and Y are the two critical values of F_ϕ, and the equations involving X and Y are symmetric in terms of the marked points x and y: $F_\phi(\omega_1) = X$, and $F_\phi(\omega_2) = Y$. We explore this further in chapter 10.

Corollary 5.2.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Then consider the monic polynomial associated to F_ϕ, $\widetilde{F_{\mathbf{w}}}$.*

$$\widetilde{F_{\mathbf{w}}}(z) = z^d + \frac{p_{d-1}(\mathbf{w})}{q_{d-1}(\mathbf{w})} z^{d-1} + \ldots + \frac{p_0(\mathbf{w})}{q_0(\mathbf{w})}.$$

Suppose that the critical points of $F_{\mathbf{w}}$ are all of the form W_i where $W_i : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}$ is a rational function. Then for all i, $W_i(\mathbf{w})$ is homogeneous of degree 1; that is

$$W_i(\mathbf{w}) = s_i(\mathbf{w})/t_i(\mathbf{w})$$

where s_i and t_i are homogeneous polynomials, and $\deg(s_i) = \deg(t_i) + 1$.

Proof. Let $\mathbf{w} \in \mathbb{C}^{n+1}$for which $\widetilde{F_{\mathbf{w}}}$ exists:

$$\widetilde{F_{\mathbf{w}}}(z) = z^d + \frac{p_{d-1}(\mathbf{w})}{q_{d-1}(\mathbf{w})} z^{d-1} + \ldots + \frac{p_0(\mathbf{w})}{q_0(\mathbf{w})}.$$

This polynomial induces a map $\widetilde{G_f} : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$, by evaluation. That is

$$\widetilde{F_{\mathbf{w}}}(\Omega_{\widetilde{F_{\mathbf{w}}}} \cup \{0, \infty, w_1, \ldots, w_{n+1}\}) = \{v_1, \ldots, v_{n+1}\},$$

and $\widetilde{G_f}(\mathbf{w}) = \mathbf{v}$. Suppose $W_i(\mathbf{w})$ is a critical point of $\widetilde{F_{\mathbf{w}}}$. Then there is a

$$v_i := v_i(\mathbf{w}) = \widetilde{F_{\mathbf{w}}}(W_i(\mathbf{w})),$$

or

$$v_i(\mathbf{w}) = (W_i(\mathbf{w}))^d + \frac{p_{d-1}(\mathbf{w})}{q_{d-1}(\mathbf{w})}(W_i(\mathbf{w}))^{d-1} + \ldots + \frac{p_0(\mathbf{w})}{q_0(\mathbf{w})}(W_i(\mathbf{w}))$$

but this quantity must be homogeneous, so the functions $W_i : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}$ must be homogeneous. So we may write $W_i = s_i/t_i$, where $s_i, t_i : \mathbb{C}^{n+1} \to \mathbb{C}$ are homogeneous polynomials.

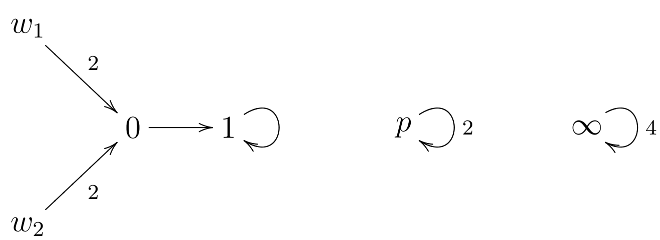
By definition, for $w_j \in \{w_1, \ldots, w_{n+1}\}$, there is a $v_j(\mathbf{w}) := \widetilde{F_{\mathbf{w}}}(w_i)$, or

$$v_j(\mathbf{w}) = (w_j)^d + \frac{p_{d-1}(\mathbf{w})}{q_{d-1}(\mathbf{w})}(w_j)^{d-1} + \ldots + \frac{p_0(\mathbf{w})}{q_0(\mathbf{w})}(w_j)$$

which is homogeneous of degree d, so since the induced map $\widetilde{G_f}$ is homogeneous, we must have that $\widetilde{F_{\mathbf{w}}}(W_i)$ is homogeneous of degree d as well, which means that $\deg(W_i) = 1$, or $\deg(s_i) = \deg(t_i) + 1$. $\square$

From the proof of theorem 4.0.1, and the corollary above, we see that if all critical points of the polynomial $\widetilde{F_{\mathbf{w}}}$ are of the form s_i/t_i, where $\deg(s_i) = \deg(t_i) + 1$, then there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. This is a sufficient condition for guaranteeing the existence of an induced map, however, we see that it is not necessary in the following example.

Example 5.2.2. Let $f : S^2 \to S^2$ be a Thurston map with the following ramification portrait.



with critical set $\Omega_f = \{w_1, w_2, p, \infty\}$, and postcritical set $P_f = \{0, 1, p, \infty\}$. Let $\tau \in \text{Teich}(S^2, \mathcal{P}_f)$ of which $\phi : (S^2, \mathcal{P}_f), \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a representative homeomorphism such that $\phi(0) = 0$, $\phi(1) = 1$, $\phi(\infty) = \infty$, and define $X := \phi(p)$. Suppose that $\tau' := \sigma_f(\tau)$ of which $\psi : (S^2, \mathcal{P}_f), \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$ is a representative homeomorphism such that $\psi(0) = 0$, $\psi(1) = 1$, $\psi(\infty) = \infty$, and define $x := \psi(p)$, $\omega_1 := \psi(w_1)$, and $\omega_2 := \psi(w_2)$. Then according to the following commutative diagram,

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F_\phi"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f))
\end{tikzcd}$$

F_ϕ is a quartic polynomial, and we have the following observations:

- the critical points of F_ϕ are ω_1, ω_2, and x
- $F_\phi(\omega_1) = 0$
- $F_\phi(\omega_2) = 0$
- $F_\phi(0) = 1$
- $F_\phi(1) = 1$
- $F_\phi(x) = X$.

A normal form for the polynomial F_ϕ is $F_\phi(z) = A(z - \omega_1)^2(z - \omega_2)^2$, so

$$F'_\phi(z) = 2A(z - \omega_1)(z - \omega_2)(2z - (\omega_1 + \omega_2)),$$

and we immediately see that

$$x = \frac{\omega_1 + \omega_2}{2} \text{ or } 2x = \omega_1 + \omega_2.$$

We rewrite F_ϕ as

$$F_\phi(z) = A\left(z^2 - 2(\omega_1 + \omega_2)z + \omega_1\omega_2\right)^2$$

and replace $\omega_1 + \omega_2$ with $2x$ to obtain

$$F_\phi(z) = A\left(z^2 - 4xz + \omega_1\omega_2\right)^2.$$

Imposing the condition that $F_\phi(0) = 1$ implies that

$$A(\omega_1\omega_2)^2 = 1 \implies A = \frac{1}{(\omega_1\omega_2)^2}.$$

Imposing the condition $F_\phi(1) = 1$ implies that

$$\begin{aligned} (\omega_1\omega_2)^2 &= (1 - 4x + \omega_1\omega_2)^2 \\ &= (1-4x)^2 + 2(1 - 4x + \omega_1\omega_2) + (\omega_1\omega_2)^2 \\ \implies \omega_1\omega_2 &= \frac{4x-1}{2}. \end{aligned}$$

So we write

$$F_\phi(z) = \frac{4}{(4x-1)^2}\left(z^2 - 4xz + \frac{4x-1}{2}\right)^2$$

so f does indeed have the $\pi\sigma$-property. We now impose the remaining condition that $X = F_\phi(x)$, which gives

$$X = \frac{(6x^2 + 1 - 4x)^2}{(4x-1)^2}.$$

This is our induced map, (written as a map on $\mathbb{C}$). Notice that we did not solve for ω_1 and ω_2 as functions of x, rather, we found the symmetric functions $\omega_1 + \omega_2$ and $\omega_1\omega_2$ as functions of x. In this case, this was sufficient to induce a map $g_f : \mathbb{P}^1 \to \mathbb{P}^1$. Notice that ω_1 and ω_2 are the roots of the equation

$$F'_\phi(z) = 2z^2 - 8xz + 4x - 1 = 0,$$

so ω_1 and ω_2 are radical functions of x.

We now address the question of whether the induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, (if it exists), is holomorphic.

Proposition 5.2.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If*

$$\text{alg deg } (g_f) = d,$$

then $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is an endomorphism.

Proof. By corollary 5.1.1, $\deg(q_i) = 0$ for all $i \in [0, d-1]$. Hence $\widetilde{F_{\mathbf{w}}}$ is defined for all $\mathbf{w} \in \mathbb{C}^{n+1}$, and thus induces a homogeneous map $\widetilde{G_f} : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$ such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{C}^{n+1} - \{\mathbf{0}\} \arrow[r, dashed, "\widetilde{G_f}"] \arrow[d] & \mathbb{C}^{n+1} \arrow[d, dashed] \\ \mathbb{P}^n \arrow[r, dashed, "g_f"] & \mathbb{P}^n \end{tikzcd}$$

Moreover, by proposition 5.1.2, $\widetilde{G_f}$ is holomorphic on $\mathbb{C}^{n+1}$. Let $\mathbf{z} \in \mathbb{P}^n$, and let $\mathbf{w} \in \mathbb{C}^{n+1} - \{\mathbf{0}\}$ be a representative of $\mathbf{z}$. The map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ has a point of indeterminacy at $\mathbf{z}$ if $\widetilde{G_f}(\mathbf{w}) = \mathbf{0}$.

Suppose there is such an $\mathbf{w} \in \mathbb{C}^{n+1} - \{\mathbf{0}\}$. We then draw the following consequences about the polynomial $\widetilde{F_{\mathbf{w}}}$.

- $\widetilde{F_{\mathbf{w}}}$ is a monic polynomial of degree d
- $\widetilde{F_{\mathbf{w}}}$ has exactly two critical values: 0 and ∞.

Just as in the proof of theorem 4.0.1, we use lemma 4.0.1 to conclude that

$$\widetilde{F_{\mathbf{w}}}(z) = z^d.$$

However, this immediately implies that $w_1, w_2, \ldots, w_{n+1} = 0$, so we see that

$$\widetilde{G_f}^{-1}(\mathbf{0}) = \{\mathbf{0}\},$$

and the map $g_f : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism. □

We now present sufficient conditions for which the induced map is holomorphic. This will be discussed further in chapter 10.

Proposition 5.2.2. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If $d < n$, and if $g_f : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism, then*

$$\text{alg deg }(g_f) = d.$$

Proof. Let $\widetilde{F_{\mathbf{w}}}$ be the monic polynomial associated to F_ϕ, and let

$$\widetilde{G_f} : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$$

be the map induced by $\widetilde{F_{\mathbf{w}}}$. Suppose that alg deg $(g_f) > d$. Then by proposition 5.1.2, $\widetilde{G_f}$ is not holomorphic. Write

$$\widetilde{G_f}(\mathbf{w}) = (v_1(\mathbf{w}), \ldots, v_{n+1}(\mathbf{w}))\,,$$

where $v_i(\mathbf{w})$ is homogeneous of degree d. The map $\widetilde{G_f}$ is not holomorphic, so there exists a nonconstant homogeneous polynomial $p(\mathbf{w})$ of minimal degree such that the map

$$H(\mathbf{w}) := (p(\mathbf{w}) \cdot v_1(\mathbf{w}), \ldots, p(\mathbf{w}) \cdot v_{n+1}(\mathbf{w}))\,,$$

is holomorphic. The algebraic degree of g_f is equal to $d + \deg(p)$. Consider the polynomial

$$h_{\mathbf{w}}(z) := p(\mathbf{w}) \cdot \widetilde{F_{\mathbf{w}}}(z).$$

This polynomial is homogeneous in the variables w_i and z, of degree $d + \deg(p)$. Moreover, this polynomial induces the map $H : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, and the following diagram commutes.

$$\begin{tikzcd}
\mathbb{C}^{n+1} - \{\mathbf{0}\} \arrow[r, "H"] \arrow[d] & \mathbb{C}^{n+1} \arrow[d, dashed] \\
\mathbb{P}^n \arrow[r, dashed, "g_f"] & \mathbb{P}^n
\end{tikzcd}$$

We may express the polynomial $h_{\mathbf{w}}$ as

$$h_{\mathbf{w}}(z) = p(\mathbf{w})z^d + \alpha_{d-1}(\mathbf{w})z^{d-1} + \ldots + \alpha_0(\mathbf{w})$$

where $\alpha_i : \mathbb{C}^{n+1} \to \mathbb{C}$ is a homogeneous polynomial of degree greater than 0.

Consider the set of equations $\alpha_i(\mathbf{w}) = 0$ for all $i \in [0, d-1]$, and the equation $p(\mathbf{w}) = 0$. This is a family of $d+1$ equations, each of which is a homogeneous polynomial in the w_i. We can therefore consider the intersection of the loci defined by each hypersurface $\alpha_i(\mathbf{w}) = 0$, and $p(\mathbf{w}) = 0$ inside $\mathbb{P}^n$. Suppose that these $d+1$ equations have a common zero in $\mathbb{P}^n$; suppose that the point $\mathbf{z}$ is such a point. Then $\mathbf{z}$ is necessarily a point of indeterminacy for the map g_f, for if $\mathbf{w} \in \mathbb{C}^{n+1}$ if any representative of $\mathbf{z}$, we have $\alpha_i(\mathbf{w}) = 0$, and $p(\mathbf{w}) = 0$, so the polynomial $h_{\mathbf{w}}$ would be identically 0 for such a $\mathbf{w}$. In terms of the induced map,

$$H : (w_1, \ldots, w_{n+1}) \longmapsto (0, \ldots, 0)$$

and $\mathbf{z}$ is a point of indeterminacy of g_f.

We must now determine if there is a nonempty intersection of the family of hypersurfaces defined by α_i and p. This is guaranteed by the hypothesis that $d < n$, for each hypersurface defines a locus in $\mathbb{P}^n$ which is of codimension 1, and so the intersection of a family of $d+1$ hypersurfaces defines a locus in $\mathbb{P}^n$ which is of codimension $d+1$. Hence, by the projective intersection theorem in [17], these hyperplanes have an intersection of dimension $n-(d+1)$, and there is a nonempty intersection if $n \geq d+1$, or if $d < n$.

Therefore, if $d < n$, and if the algebraic degree of g_f is greater than d then there are points of indeterminacy of the map g_f.

$\square$

Propositions 5.2.1 and 5.2.2 prove part of the following conjecture.

Conjecture 5.2.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Then the map $g_f : \mathbb{P}^n \to \mathbb{P}^n$ is holomorphic if and only if*

$$\text{alg deg }(g_f) = d.$$

Proposition 5.2.1 proves the 'if' direction of the conjecture, and proposition 5.2.2 proves the 'only if' direction, provided that $d < n$. We discuss this conjecture again in chapter 10.

CHAPTER 6

CONSEQUENCES FOR THE THURSTON PULLBACK MAP

Proposition 6.0.3. *Let f be a Thurston map with postcritical set $\mathcal{P}_f$. Identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with $\mathbb{P}^n - \Delta$ in the standard way. Suppose there is an endomorphism $g : \mathbb{P}^n \to \mathbb{P}^n$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"'] \end{tikzcd}$$

Then the image of σ_f is open in $\mathrm{Teich}(S^2, \mathcal{P}_f)$.

Proof. The proof of this follows from the commutative diagram involved. We first manufacture a one-sided inverse for the map $g : \mathbb{P}^n \to \mathbb{P}^n$. Let $\mathbf{y} \in \mathbb{P}^n - \Delta$, and choose a neighborhood $U_{\mathbf{y}}$ of $\mathbf{y}$. Let $\tau_{\mathbf{y}} \in \mathrm{Teich}(S^2, \mathcal{P}_f)$ be any lift of $\mathbf{y}$, and let $V_{\tau_{\mathbf{y}}}$ be a lift of $U_{\mathbf{y}}$ in the fundamental domain which contains $\tau_{\mathbf{y}}$; let $\rho : (U_{\mathbf{y}}, \mathbf{y}) \to (V_{\tau_{\mathbf{y}}}, \tau_{\mathbf{y}})$ denote the appropriate branch of π^{-1}. Define the map $q_{\mathbf{y}}$ so that the following diagram commutes.

$$\begin{tikzcd} \left(V_{\tau_{\mathbf{y}}}, \tau_{\mathbf{y}}\right) \arrow[r, "\sigma_f"] & \left(\sigma_f(V_{\tau_{\mathbf{y}}}), \sigma_f(\tau_{\mathbf{y}})\right) \arrow[d, "\pi"] \\ (U_{\mathbf{y}}, \mathbf{y}) \arrow[u, "\rho"] \arrow[r, "q_{\mathbf{y}}"] & \left(\pi\left(\sigma_f(V_{\tau_{\mathbf{y}}})\right), \pi\left(\sigma_f(\tau_{\mathbf{y}})\right)\right) \end{tikzcd}$$

Note that

$$g(\pi(\sigma_f(\tau_{\mathbf{y}}))) = \mathbf{y},$$

and so the map

$$q_{\mathbf{y}} : (U_{\mathbf{y}}, \mathbf{y}) \longrightarrow \left(\pi\left(\sigma_f(V_{\tau_{\mathbf{y}}})\right), \pi\left(\sigma_f(\tau_{\mathbf{y}})\right)\right)$$

is a one-sided inverse for g. We have

$$g \circ q_{\mathbf{y}} : U_{\mathbf{y}} \longrightarrow \pi\left(\sigma_f\left(V_{\tau_{\mathbf{y}}}\right)\right) \qquad \text{is} \qquad \mathrm{Id}|_{U_{\mathbf{y}}} : U_{\mathbf{y}} \longrightarrow U_{\mathbf{y}}.$$

By the chain rule, we see that $D\sigma_f|_{\tau_{\mathbf{y}}}$ is therefore invertible in a neighborhood of $\tau_{\mathbf{y}} \in \mathrm{Teich}(S^2, \mathcal{P}_f)$. Since the choice of $\mathbf{y} \in \mathbb{P}^n - \Delta$ was arbitrary, and the choice of lift $\tau_{\mathbf{y}}$ was also arbitrary, we see that $D\sigma_f|_\tau$ is invertible at every element $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$. It follows from the inverse function theorem that $\sigma_f\left(\mathrm{Teich}(S^2, \mathcal{P}_f)\right)$ is open in $\mathrm{Teich}(S^2, \mathcal{P}_f)$. □

Corollary 6.0.2. *Let f be a Thurston map with postcritical set $\mathcal{P}_f$. Identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with $\mathbb{P}^n - \Delta$ in the standard way. Suppose there is an endomorphism $g : \mathbb{P}^n \to \mathbb{P}^n$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"']
\end{tikzcd}$$

Then the $g : \mathbb{P}^n \to \mathbb{P}^n$ is unique.

Proof. Suppose that there are two such endomorphisms $g_1, g_2 : \mathbb{P}^n \to \mathbb{P}^n$, for which the diagram above commutes. As proved in proposition 6.0.3

$$\sigma_f\left(\mathrm{Teich}(S^2, \mathcal{P}_f)\right) \subseteq \mathrm{Teich}(S^2, \mathcal{P}_f) \text{ is open,}$$

so

$$W_f := \pi\left(\sigma_f\left(\mathrm{Teich}(S^2, \mathcal{P}_f)\right)\right) \subseteq \mathbb{P}^n - \Delta$$

is also open in the topology that $\mathbb{P}^n$ inherits as a complex manifold. From the commutative diagram, we have

$$g_1 \circ \pi \circ \sigma_f = \pi \qquad \text{and} \qquad g_2 \circ \pi \circ \sigma_f = \pi$$

which implies that $g_1|_{W_f} = g_2|_{W_f}$. Since g_1 and g_2 are endomorphisms of $\mathbb{P}^n$, we consider the set $A := \{z \in \mathbb{P}^n \mid g_1(z) = g_2(z)\}$. This set is algebraic, and so it is closed in the Zariski topology of $\mathbb{P}^n$. Note that $W_f \subseteq A$, and so $\operatorname{codim}(A) = 0$. Since $\mathbb{P}^n$ is *irreducible* with respect to the Zariski topology (see [31]), we must have that $\mathbb{P}^n = A$, so $g_1 = g_2$ as endomorphisms of $\mathbb{P}^n$. $\square$

Proposition 6.0.4. *Let f be a Thurston map with postcritical set $\mathcal{P}_f$. Identify $\operatorname{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose there is an endomorphism $g : \mathbb{P}^n \to \mathbb{P}^n$ such that the following diagram commutes.*

$$\begin{tikzcd}
\operatorname{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \operatorname{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"']
\end{tikzcd}$$

Define $\mathcal{L} := g^{-1}(\Delta)$. Then $\sigma_f\left(\operatorname{Teich}(S^2, \mathcal{P}_f)\right) \subseteq \operatorname{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L})$.

Proof. We proceed by contradiction. Suppose there is $\tau \in \operatorname{Teich}(S^2, \mathcal{P}_f)$ such that $\sigma_f(\tau) \in \pi^{-1}(\mathcal{L})$. Then

$$\pi\left(\sigma_f(\tau)\right) \in \mathcal{L} \implies g\left(\pi\left(\sigma_f(\tau)\right)\right) \in \Delta,$$

but the commutative diagram implies that

$$g\left(\pi\left(\sigma_f(\tau)\right)\right) = \pi(\tau)$$

so $\pi(\tau) \in \Delta$, which is a contradiction. $\square$

Proposition 6.0.5. *Let f be a Thurston map with postcritical set $\mathcal{P}_f$. Identify $\operatorname{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose there is an endomorphism $g : \mathbb{P}^n \to \mathbb{P}^n$ which makes the following diagram commute.*

$$\begin{tikzcd}
\operatorname{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \operatorname{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"']
\end{tikzcd}$$

Suppose also that the critical value locus of g is contained in Δ. Then

1. $\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))$ *is a covering map, and*

2. $\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f)) = \mathrm{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L})$.

Proof. Since $g : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism whose critical values are contained in Δ,

$$g : \mathbb{P}^n - \mathcal{L} \longrightarrow \mathbb{P}^n - \Delta$$

is a covering map since it is a local homeomorphism, and it is proper (see p. 23 of [9]). Therefore, the composition

$$g \circ \pi : \mathrm{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L}) \longrightarrow \mathbb{P}^n - \Delta$$

is a covering map as well. Let $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$, and $\tau' = \sigma_f(\tau)$. Since the diagram in the hypothesis of the proposition commutes, we have

$$\pi(\tau) = g(\pi(\tau')).$$

We have the following two covering spaces:

$$\begin{tikzcd}
(\mathrm{Teich}(S^2, \mathcal{P}_f), \tau) \arrow[d, "\pi"'] & & (\mathrm{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L}), \tau') \arrow[dll, "g\circ\pi"] \\
(\mathbb{P}^n - \Delta, \pi(\tau)) & &
\end{tikzcd}$$

Since $\pi : (\mathrm{Teich}(S^2, \mathcal{P}_f), \tau) \longrightarrow (\mathbb{P}^n - \Delta, \pi(\tau))$ is a universal cover, there is a unique lift $\sigma : (\mathrm{Teich}(S^2, \mathcal{P}_f), \tau) \longrightarrow (\mathrm{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L}), \tau')$ such that this diagram commutes.

$$\begin{tikzcd}
(\mathrm{Teich}(S^2, \mathcal{P}_f), \tau) \arrow[r, "\sigma"] \arrow[d, "\pi"'] & (\mathrm{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L}), \tau') \arrow[dl, "g \circ \pi"] \\
(\mathbb{P}^n - \Delta, \pi(\tau)) &
\end{tikzcd}$$

Moreover, $\sigma : (\mathrm{Teich}(S^2, \mathcal{P}_f), \tau) \longrightarrow (\mathrm{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L}), \tau')$ is a covering map as well. By uniqueness, we must have $\sigma = \sigma_f$, and we have proven the proposition as it immediately follows from the arguments above that

$$\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f)) = \mathrm{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L}).$$

$\square$

We now prove that if the Thurston map f is a topological polynomial of degree d, such there is an induced map $g_f : \mathbb{P}^n \to \mathbb{P}^n$ of algebraic degree d, then it is necessarily postcritically finite. First we require the following topological fact, stated as a lemma.

Lemma 6.0.1. *Let D_i be a closed topological disk with i punctures. Suppose that $F : D_n \to D_m$ is a covering map of degree d. Then we must have $n-1 = d(m-1)$, and in particular, $m = 1 \iff n = 1$.*

Proof. This is a standard fact from topology, and we therefore omit the proof.

$\square$

The lemma above proves that the only finite-sheeted covering space of a once punctured disk is a finite union of once punctured disks.

Proposition 6.0.6. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If*

$$\text{alg deg}(g_f) = d,$$

then $g_f(\Delta) \subseteq \Delta$.

Proof. Since alg deg $(g_f) = d$, we have that

- the monic polynomial $\widetilde{F_{\mathbf{w}}}$ is defined for all $\mathbf{w} \in \mathbb{C}^{n+1}$,
- it induces a holomorphic map $\widetilde{G_f} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, and gives the map g_f in homogeneous coordinates,
- $g_f : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism.

We now proceed with the proof of the proposition, which is based primarily on the Mumford compactness theorem, the Grötzsch inequality and the subadditivity of annuli (see [18]).

Recall that we have identified $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$. For $N > 0$ let $\mathcal{M}_N$ denote the subset of $\mathbb{P}^n - \Delta$ consisting of Riemann surfaces containing an essential nonperipheral annulus of modulus larger than N; that is, a point $\mathbf{x} \in \mathbb{P}^n - \Delta$ corresponds to the Riemann sphere $\mathbb{P}^1$ with $n + 3$ punctures.

Fix $\mathbf{x} \in \mathbb{P}^n - (\Delta \cup \mathcal{L})$, and consider $\mathbf{y} := g_f(\mathbf{x})$. Choose a $\mathbf{w} \in \mathbb{C}^{n+1} - \{\mathbf{0}\}$, which represents $\mathbf{x}$, and define $\mathbf{v} := \widetilde{G_f}(\mathbf{w})$, which represents $\mathbf{y}$. Recall that $\widetilde{F_{\mathbf{w}}}$ induces the map $\widetilde{G_f}$ by evaluation:

$$\widetilde{F_{\mathbf{w}}} : (\mathbb{P}^1, \{0, \infty, w_1, \ldots, w_{n+1}\}) \longrightarrow (\mathbb{P}^1, \{0, \infty, v_1, \ldots, v_{n+1}\}).$$

Let $X_{\mathbf{w}}$ be the Riemann sphere marked with the set $\{0, \infty, w_1, \ldots, w_{n+1}\}$, and let $Y_{\mathbf{v}}$ be the Riemann sphere marked with the set $\{0, \infty, v_1, \ldots, v_{n+1}\}$; then $\widetilde{F_{\mathbf{w}}}$ maps $X_{\mathbf{w}}$ to $Y_{\mathbf{v}}$.

Let $X'_{\mathbf{w}}$ be $X_{\mathbf{w}} - \widetilde{F_{\mathbf{w}}}^{-1}(\{0, \infty, v_1, \ldots, v_{n+1}\})$. Notice that $\widetilde{F_{\mathbf{w}}} : X'_{\mathbf{w}} \to Y_{\mathbf{v}}$ is a covering map of degree d.

Let $\mathbf{z} \in \Delta$, and choose $\mathbf{x} \in \mathbb{P}^n - (\Delta \cup \mathcal{L})$ close to $\mathbf{z}$. Then by the Mumford compactness theorem, $\mathbf{x} \in \mathcal{M}_N$ for some N very large. Thus there is a nonperipheral, essential annulus $A \subset X_{\mathbf{w}}$ of modulus larger than N. At the expense of an additive constant, we may suppose that A is a Euclidean annulus (a right cylinder).

Consider $A' = A \cap X'_{\mathbf{w}}$. This surface consists of A with at most $d|\mathcal{P}_f|$ points removed, so there is a Euclidean subannulus $\widetilde{B} \subset A'$ whose modulus is at least

$$\frac{N}{d|\mathcal{P}_f|}.$$

Let γ be the geodesic in the homotopy class of the core curves of A'. Since A' is of large modulus, γ is very short. Note that γ is essential, and nonperipheral. Consider $F_{\mathbf{w}}(\gamma)$. Since $F_{\mathbf{w}}$ is a covering map, it is a local isometry for the Poincaré metrics on $X'_{\mathbf{w}}$ and $Y_{\mathbf{v}}$. Hence, the length of $F_{\mathbf{w}}(\gamma)$ is less than or equal to the length of γ. Moreover, as $N \to \infty$, the length of γ must tend to 0, hence the length of $F_{\mathbf{w}}(\gamma)$ must tend to 0 also, so the curve $F_{\mathbf{w}}(\gamma)$ cannot intersect itself (see [10]); so $F_{\mathbf{w}}(\gamma)$ is homotopic to a simple closed curve. Let $\delta \subset Y_{\mathbf{v}}$ be the geodesic in this homotopy class. The curve δ must be essential and nonperipheral, for otherwise, γ would not have been essential and nonperipheral (see lemma 6.0.1).

As $\mathbf{x} \to \mathbf{z}$, $N \to \infty$, and so the length of $\gamma \subset X'_{\mathbf{w}}$ tends to 0, and the length of $\delta \subset Y_{\mathbf{v}}$ tends to 0 as well as $\mathbf{x}$ tends to $\mathbf{z}$. Moreover, as $\mathbf{x} \to \mathbf{z}$, $\mathbf{y} \to g(\mathbf{z})$ by continuity, and since γ is nonperipheral and essential, we must have $g(\mathbf{z}) \in \Delta$. $\square$

Remark 6.0.1. We saw from the above proof that $\Delta \subset \mathcal{L}$. We make a point now to mention that $\mathcal{L} \neq \Delta$. If $\mathcal{L} = \Delta$, then Δ is an exceptional set for g_f. Note that proposition 3.2.1 asserts that Δ is composed of $(n+1)(n+2)/2$ hyperplanes. According to proposition 4.2 of [14], an exceptional set in $\mathbb{P}^n$ can have at most $n+1$ components, so we see that Δ is not exceptional. Hence, Δ is a proper subset of $\mathcal{L}$. Notice that proposition 6.0.5 proves that $\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f)$ is not surjective.

Lemma 6.0.2. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Suppose that* alg deg $(g_f) = d$.

Let $\mathbf{x} \in \mathbb{P}^n - (\mathcal{L} \cup \Delta)$ and let $\mathbf{y} := g_f(\mathbf{x})$. Then:

- *there exists a Thurston map $f' : S^2 \to S^2$ with postcritical set $\mathcal{P}_f$, such that R_f and $R_{f'}$ are isomorphic,*
- *there is $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$ such that $\pi(\tau) = \mathbf{y}$ and $\pi(\sigma_{f'}(\tau)) = \mathbf{x}$,*
- *and the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_{f'}"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"'] \end{tikzcd}$$

Proof. Since alg deg $(g_f) = d$, the map $\widetilde{F_{\mathbf{w}}}$ induces a holomorphic map

$$\widetilde{G_f} : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}.$$

Let $\mathbf{x}, \mathbf{y}$ be as in the hypothesis. Since $\mathbf{x}, \mathbf{y} \in \mathbb{P}^n - \Delta$, we may write

$$\mathbf{x} = [x_1 : \ldots : x_n : 1] \text{ and } \mathbf{y} = [y_1 : \ldots : y_n : 1].$$

Let $\mathbf{w} = (x_1, \ldots, x_n, 1)$, and define $\mathbf{v} := \widetilde{G_f}(\mathbf{w})$. Then $\mathbf{v} \in \mathbb{C}^{n+1}$ represents $\mathbf{y} \in \mathbb{P}^n$. Let $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ be a homeomorphism normalized so that $\phi(p_{n+2}) = 0, \phi(\infty) = \infty$, and $\phi(p_i) = v_i$ for $i \in [1, n+1]$, and let the homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$ be normalized so that $\psi(p_{n+2}) = 0$, and $\psi(\infty) = \infty$, and that $\psi(p_i) = w_i$ for $i \in [1, n+1]$.

Define the map $f' := \phi^{-1} \circ \widetilde{F_{\mathbf{w}}} \circ \psi$. This is evidently a Thurston map with postcritical set $\mathcal{P}_f$. Moreover, $R_{f'}$ is isomorphic to R_f, and we have the following commutative diagram:

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f'"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "\widetilde{F_{\mathbf{w}}}"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f))
\end{tikzcd}$$

by design. Observe that f' is a topological polynomial of degree d. Let the element $\tau \in \text{Teich}(S^2, \mathcal{P}_f)$ be the equivalence class of ϕ, and let $\tau' \in \text{Teich}(S^2, \mathcal{P}_f)$ be the equivalence class of ψ. Notice that $\tau' = \sigma_{f'}(\tau)$, and that $\pi(\tau) = \mathbf{y}$ and $\pi(\tau') = \mathbf{x}$.

By construction, the following diagram commutes:

$$\begin{tikzcd}
\text{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma'_f"] \arrow[d, "\pi"'] & \text{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"']
\end{tikzcd}$$

This is discussed in more detail in chapter 10. $\square$

Proposition 6.0.7. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify* $\text{Mod}(S^2, \mathcal{P}_f)$ *with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, such that* alg deg $(g_f) = d$. *Then the critical locus of g_f is contained in $\mathcal{L}$.*

Proof. By proposition 6.0.6, $\Delta \subset \mathcal{L}$. Suppose that there is $\mathbf{x} \in \mathbb{P}^n - \mathcal{L}$ which is contained in the critical locus of g_f. Then let $\mathbf{y} = g_f(\mathbf{x}) \notin \Delta$. Using lemma 6.0.2, we manufacture a Thurston map f' such that the following diagram commutes,

$$\begin{tikzcd} \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_{f'}"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"'] \end{tikzcd}$$

and we have a τ and τ' such that $\sigma_{f'}(\tau) = \tau'$, and $\pi(\tau) = \mathbf{y}$ and $\pi(\tau') = \mathbf{x}$.

Since the diagram commutes,

$$g_f \circ \pi \circ \sigma_{f'} = \pi,$$

and in particular, this identity holds in a neighborhood of $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$. Recall however, that $\mathbf{x} = \pi(\sigma_{f'}(\tau))$ is a critical point of g_f, but π has no critical points, therefore, the critical point $\mathbf{x}$ must belong to $\mathcal{L}$. $\square$

Corollary 6.0.3. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, such that* $\mathrm{alg\ deg}\ (g_f) = d$. *Then $g_f : \mathbb{P}^n \to \mathbb{P}^n$ is a postcritically finite endomorphism.*

Proof. Since $\mathrm{alg\ deg}\ (g_f) = d$, g_f is an endomorphism. Since the critical locus of g_f is contained in $\mathcal{L}$ by proposition 6.0.7, the critical value locus of g_f is necessarily contained in Δ. It then follows from proposition 6.0.6 that g_f is postcritically finite. $\square$

We summarize the results of this chapter with the following theorem.

Theorem 6.0.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with $\mathbb{P}^n - \Delta$ in he standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, such that* $\mathrm{alg\ deg}\,(g_f) = d$. *Then:*

1. *g_f is a postcritically finite endomorphism, and*
2. *$\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f)) = \mathrm{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L})$, and*
3. *$\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f) - \pi^{-1}(\mathcal{L})$ is a covering map.*

The pullback map $\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f)$ has gotten some attention recently. In fact, the results above inspired the following theorem, which is proven in [6].

Theorem 6.0.2 (Buff, Epstein, Koch, Pilgrim)**.** *There exist Thurston maps f for which σ_f is contracting, has a fixed point τ and:*

1. *the derivative of σ_f is invertible at τ, the image of σ_f is open and dense in* $\mathrm{Teich}(S^2, \mathcal{P}_f)$ *and $\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \sigma_f\big(\mathrm{Teich}(S^2, \mathcal{P}_f)\big)$ is a covering map,*
2. *the derivative of σ_f is not invertible at τ, the image of σ_f is equal to* $\mathrm{Teich}(S^2, \mathcal{P}_f)$ *and $\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f)$ is a ramified Galois covering map, or*
3. *the map σ_f is constant.*

CHAPTER 7

THE AUGMENTED TEICHMÜLLER SPACE

Let f be a Thurston map for which there exists a postcritically finite endomorphsim $g : \mathbb{P}^n \to \mathbb{P}^n$ such that the diagram

$$\begin{tikzcd} \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"'] \end{tikzcd}$$

commutes. Since g is defined on all of $\mathbb{P}^n$ and not just on the "moduli space" part of $\mathbb{P}^n$ (not just on $\pi(\mathrm{Teich}(S^2, \mathcal{P}_f)) = \mathbb{P}^n - \Delta$), we would like to discuss a boundary of $\mathrm{Teich}(S^2, \mathcal{P}_f)$ which corresponds to "surfaces with nodes." Adding this boundary to the Teichmüller space gives us a new space, $\overline{\mathrm{Teich}(S^2, \mathcal{P}_f)}$, called the *augmented Teichmüller space*. This space was introduced in 1977 by Abikoff in [1]. We present the definition of the augmented Teichmüller space for a compact oriented surface S of genus g, and a finite subset Z. We then discuss this for the case where $g = 0$, and Z is the postcritical set of a Thurston map f. The following discussion of the augmented Teichmüller space is extracted from [20].

Let S be a compact, oriented surface of genus g, and $Z \subset S$ be a finite set, with n points, where $2g - 2 + n \geq 0$. We define $\overline{\mathrm{Teich}(S, Z)}$ in the following way.

Definition 7.0.1. *The* augmented Teichmüller space of (S, Z)*,* $\overline{\mathrm{Teich}(S, Z)}$*, is the set of analytic curves, which are smooth except for ordinary double points, together with a map*

$$\phi : (S, Z) \longrightarrow (X, \phi(Z))$$

where ϕ is a homeomorphism from $S/\Gamma \to X$, where Γ is a multicurve on $S - Z$, and S/Γ is defined to be S where each component of Γ has collapsed to a point, *modulo an equivalence relation* $\sim$*:*

$\phi_1 : S \to X_1$ *and* $\phi_2 : S \to X_2$ *are* $\sim$*-equivalent if and only if there exists a complex analytic isomorphism* $\alpha : (X_1, \phi_1(Z)) \longrightarrow (X_2, \phi_2(Z))$*, a homeomorphism* $\beta : (S, Z) \longrightarrow (S, Z)$*, which is the identity on* Z*, and which is isotopic to the identity relative to* Z *such that:*

$$\begin{array}{ccc} (S,Z) & \xrightarrow{\phi_1} & (X_1, \phi_1(Z)) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ (S,Z) & \xrightarrow{\phi_2} & (X_2, \phi_2(Z)) \end{array}$$

commutes, and

$$\alpha \circ \phi_1|_Z = \phi_2|_Z.$$

We now discuss the topology of the $\overline{\text{Teich}(S,Z)}$. An ϵ-neighborhood

$$U_\epsilon \subset \overline{\text{Teich}(S,Z)}$$

of the homeomorphism $\phi : S/\Gamma \to X$ consists of $\phi_1 : S/\Gamma_1 \to X_1$ such that

- $\Gamma_1 \subseteq \Gamma$ up to homotopy
- the geodesics in the homotopy classes of $\phi_1(\gamma), \gamma \in \Gamma - \Gamma_1$ are short (they all have length less than ϵ),
- there exists a $1 + \epsilon$ quasiconformal map

  $$\alpha : (X_1 - \phi_1(Z)) - A_\Gamma(X_1 - \phi_1(Z)) \longrightarrow (X - \phi(Z)) - A_\Gamma(X - \phi(Z))$$

  where A_Γ is the collection of "standard collar" annuli about the curves of Γ (see the collaring theorem in [18]).

Remark 7.0.2. The boundary of $\overline{\mathrm{Teich}(S,Z)}$ is composed of strata $\mathcal{S}_\Gamma$, corresponding to the multicurve Γ collapsing. Each one of these strata is naturally a Teichmüller space of smaller dimension; it is the Teichmüller space of the surface S/Γ with punctures at the points corresponding to the each component of Γ that collapsed. In particular, the minimal strata, corresponding to maximal multicurves, are points.

Here is an example illustrating how complicated $\overline{\mathrm{Teich}(S,Z)}$ can be.

Example 7.0.3. In the case where $g = 1$, and $|Z| = 1$, $\overline{\mathrm{Teich}(S,Z)}$ can be identified with $\mathbb{H}^+ \cup \mathbb{P}^1(\mathbb{Q})$, where $\mathbb{P}^1(\mathbb{Q})$ is the projective line over the rational numbers, which is just $\mathbb{Q} \cup \{\infty\}$. A neighborhood of a rational number $q \in \mathbb{Q}$ is the union of q with a horodisc based at q.

7.1 The augmented moduli space $\overline{\mathrm{Mod}(S,Z)}$

Remark 7.1.1. The mapping class group of (S,Z) acts on $\overline{\mathrm{Teich}(S,Z)}$ by homeomorphisms: for f representing an element $[f] \in \mathrm{MCG}(S,Z)$, the action is given by $f \cdot (X,\phi) := (X, \phi \circ f)$.

Since the action of the mapping class group extends to $\overline{\mathrm{Mod}(S,Z)}$, we can define the quotient

$$\overline{\mathrm{Mod}(S,Z)} := \overline{\mathrm{Teich}(S,Z)}/\mathrm{MCG}(S,Z)$$

which we call the *augmented moduli space*.

Definition 7.1.1. *The quotient* $\overline{\mathrm{Teich}(S,Z)}/\mathrm{MCG}(S,Z)$ *is the* augmented moduli space of (S,Z).

Viewed as just a topological space, $\overline{\mathrm{Mod}(S,Z)}$ is compact and normal. However, it is actually a complex analytic space (see [20]). The following result is often cited in the literature, but the proof is much more elusive. A complete proof can be found in [20].

Theorem 7.1.1. *In the category of complex analytic spaces, the augmented moduli space is isomorphic to the Deligne-Mumford compactification of* $\mathrm{Mod}(S,Z)$*, that is,* $\overline{\mathrm{Mod}(S,Z)}$ *is isomorphic to* $\overline{\mathrm{Mod}(S,Z)}_{\mathrm{DM}}$.

7.2 The Weil-Petersson metric completion of $\mathrm{Teich}(S,Z)$

From some perspectives, the Teichmüller metric is the most natural metric for $\mathrm{Teich}(S,Z)$. For example, in the case where $g=0$, and where $Z=\mathcal{P}_f$, our surface is the topological 2-sphere, and for any Thurston map $f:S^2\to S^2$, the pullback map

$$\sigma_f:\mathrm{Teich}(S^2,\mathcal{P}_f)\longrightarrow \mathrm{Teich}(S^2,\mathcal{P}_f)$$

is weakly contracting in the Teichmüller metric. The Thurston map f is equivalent to a rational function if and only if σ_f has a fixed point $\tau\in\mathrm{Teich}(S^2,\mathcal{P}_f)$; in this case, the fixed point is unique (see [10]).

However, there is another metric on $\mathrm{Teich}(S,Z)$ which is also useful in the present context. This metric is called the Weil-Petersson metric. For a discussion of the Teichmüller and Weil-Petersson metrics, please see [35], and [18].

Inspired by work of H. Masur in [28], S. Wolpert proved that the Weil-Petersson metric completion of $\mathrm{Teich}(S,Z)$ is homeomorphic to $\overline{\mathrm{Teich}(S,Z)}$ in [35]. We paraphrase his result in the following theorem.

Theorem 7.2.1 (Masur, Wolpert)**.** *The identity map extends as a homeomorphism from the Weil-Petersson completion of* $\mathrm{Teich}(S, Z)$ *to* $\overline{\mathrm{Teich}(S, Z)}$*.*

7.2.1 The case where $g = 0$ and $Z = \mathcal{P}_f$

We now consider the case where $g = 0$, and $Z = \mathcal{P}_f$, so our surface S is the topological 2-sphere. Let $f : S^2 \to S^2$ be a Thurston map of topological degree d, with postcritical set $\mathcal{P}_f$, and pullback map $\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f)$. Using theorem 7.2.1, N. Selinger proves the following result in [34].

Theorem 7.2.2 (Selinger)**.** *The pullback map*

$$\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f)$$

extends to a continuous map $\sigma_f : \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \longrightarrow \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)}$*, and this map is Lipschitz in the Weil-Petersson metric, with Lipschitz constant* $\sqrt{d}$*.*

So theorems 7.1.1 and 7.2.2 imply that for any Thurston map $f : S^2 \to S^2$, we have the following diagram:

$$\begin{array}{ccc}
\overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} & \xrightarrow{\sigma_f} & \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \\
\Big\downarrow{\scriptstyle \overline{\pi}} & & \Big\downarrow{\scriptstyle \overline{\pi}} \\
\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}} & & \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}}
\end{array}$$

where $\overline{\pi} : \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \longrightarrow \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}$ is the quotient map representing the action of the pure mapping class group. Choose a normalization, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$. By theorem 3.3.1, there is a "blow-down" map

$$\beta : \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}} \to \mathbb{P}^n,$$

where $n = |\mathcal{P}_f| - 3$. Define the map

$$\tilde{\pi} := \beta \circ \overline{\pi} : \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \longrightarrow \mathbb{P}^n,$$

so we now have the following diagram:

$$\begin{tikzcd} \overline{\mathrm{Teich}(S^2,\mathcal{P}_f)} \arrow[r, "\sigma_f"] \arrow[d, "\tilde{\pi}"'] & \overline{\mathrm{Teich}(S^2,\mathcal{P}_f)} \arrow[d, "\tilde{\pi}"] \\ \mathbb{P}^n & \mathbb{P}^n \end{tikzcd}$$

Proposition 7.2.1. *Suppose there is an endomorphism $g : \mathbb{P}^n \to \mathbb{P}^n$ such that the following diagram commutes*

$$\begin{tikzcd} \mathrm{Teich}(S^2,\mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2,\mathcal{P}_f) \arrow[d, "\pi"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"'] \end{tikzcd}$$

then the following diagram commutes as well.

$$\begin{tikzcd} \overline{\mathrm{Teich}(S^2,\mathcal{P}_f)} \arrow[r, "\sigma_f"] \arrow[d, "\tilde{\pi}"'] & \overline{\mathrm{Teich}(S^2,\mathcal{P}_f)} \arrow[d, "\tilde{\pi}"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"'] \end{tikzcd}$$

Proof. The proof is an immediate consequence of the extension of

$$\sigma_f : \mathrm{Teich}(S^2, \mathcal{P}_f) \longrightarrow \mathrm{Teich}(S^2, \mathcal{P}_f)$$

given in [34]. ☐

Suppose $f : S^2 \to S^2$ is a Thurston map with postcritical set $\mathcal{P}_f$, and there exists an endomorphism $g : \mathbb{P}^n \to \mathbb{P}^n$ for which the following diagram commutes.

$$\begin{tikzcd}
\overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \arrow[r, "\sigma_f"] \arrow[d, "\tilde{\pi}"'] & \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \arrow[d, "\tilde{\pi}"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"']
\end{tikzcd}$$

We may naturally inquire about the existence of a holomorphic lift of g to $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}}$,

$$\tilde{g} : \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}} \longrightarrow \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}}$$

for which the two smaller rectangles of the following diagram commute (the larger rectangle commutes by proposition 7.2.1).

$$\begin{tikzcd}
\overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \arrow[r, "\sigma_f"] \arrow[d, "\overline{\pi}"'] & \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \arrow[d, "\overline{\pi}"] \\
\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}} \arrow[d, "\beta"'] & \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}} \arrow[l, "\tilde{g}"'] \arrow[d, "\beta"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"']
\end{tikzcd}$$

We address this in chapter 9.

7.3 Stratified Moduli Space

As mentioned in remark 7.0.2, the strata on the boundary of the augmented Teichmüller space correspond to Teichmüller spaces of lower dimension. The same is true of the strata of the augmented moduli space: the different strata correspond to moduli spaces of lower dimension. As previously mentioned in theorem 3.3.1, we obtain the Deligne-Mumford compactification of the moduli space as a sequential blow up of $\mathbb{P}^n$, and we have a map

$$\beta : \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}} \longrightarrow \mathbb{P}^n$$

which is the blow-down map. Note that β maps the boundary of $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}}$ to the boundary of $\mathbb{P}^n$, which is the forbidden locus, Δ. Hence, the hyperplanes of Δ correspond to moduli spaces of lower dimension; $\mathbb{P}^n$ represents a *stratified moduli space*.

Let $f : S^2 \to S^2$ be a Thurston map for which there exists a postcritically finite endomorphism $g : \mathbb{P}^n \to \mathbb{P}^n$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g"']
\end{tikzcd}$$

Note that the postcritical locus of g is contained in Δ by corollary 6.0.3. Thus some of the hyperplanes in Δ are periodic. Let Π be such a hyperplane. Suppose that Π is periodic of period N. Then

$$g^{\circ N}|_{\Pi} : \Pi \longrightarrow \Pi;$$

for notational purposes, define

$$g' := g^{\circ N}|_{\Pi},$$

and define $z_0 := 0$.

Recall that $\Delta := \{[z_1 : \ldots : z_{n+1}] \in \mathbb{P}^n \ : \ z_i = z_j \text{ for some } 0 \leq i < j \leq n+1\}$. So Π is a hyperplane of the form $z_i = z_j$, and is isomorphic to $\mathbb{P}^{n-1}$ via the following the isomorphism:

$$[z_1 \ldots : z_{j-1} : z_j : z_{j+1} : \ldots : z_{n+1}] \longmapsto [z_1 \ldots : z_{j-1} : z_{j+1} : \ldots : z_{n+1}],$$

and hence, one may naturally ask the following questions:

1. is there a Thurston map f', with postcritical set $\mathcal{P}'_f \subset \mathcal{P}_f$ such that the following diagram commutes?

$$\begin{tikzcd} \mathrm{Teich}(S^2, \mathcal{P}'_f) \arrow[r, "\sigma_{f'}"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}'_f) \arrow[d, "\pi"] \\ \mathbb{P}^{n-1} & \mathbb{P}^{n-1} \arrow[l, "g'"'] \end{tikzcd}$$

2. is $g' : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ a postcritically finite endomorphism?

It is clear that since $g : \mathbb{P}^n \to \mathbb{P}^n$ is an endomorphism, then $g' : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is also an endomorphism. We discuss whether it is postcritically finite in section 7.3.1. We first address question 1.

Proposition 7.3.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that* $\text{alg deg}\,(g_f) = d$. *Then point 1 above holds.*

Proof. Define $z_0 := 0$, and fix $\Pi \in \Delta$. There is $0 \leq i < j \leq n+1$ such that Π is the hyperplane defined by the equation $z_i = z_j$. This hyperplane corresponds to the points p_i and p_j "coalescing". Define $\mathcal{P}'_f := \mathcal{P}_f - \{p_j\}$. Define

$$\Pi_0 := [z_1 : \ldots : z_{j-1} : z_i : z_{j+1} : \ldots : z_{n+1}]$$

For $l > 0$, set Π_l to be the hyperplane $g_f^{\circ l}(\Pi_0)$. Let P_0 be the hyperplane in $\mathbb{C}^{n+1}$ defined by the equation $w_i = w_j$. Then we consider $P_0, \ldots, P_{N-1}$, and consider the composition

$$F_\Pi := \widetilde{F}_{P_{N-1}} \circ \cdots \circ \widetilde{F}_{P_0},$$

which is a monic polynomial of degree d^N.

Define $w_0 := 0$, and $t_0 := 0$. Then

$$F_\Pi : \left(\mathbb{P}^1, \{0, \infty, w_1, \ldots, w_{j-1}, w_i, w_{j+1}, \ldots, w_{n+1}\}\right) \longmapsto$$

$$\left(\mathbb{P}^1, \{0, \infty, t_1, \ldots, t_{j-1}, t_i, t_{j+1}, \ldots, t_{n+1}\}\right).$$

Consider S^2 marked with the set $\mathcal{P}'_f$, and the moduli space $\mathrm{Mod}(S^2, \mathcal{P}'_f)$. We normalize in a consistent way with that above, so that for any $\psi \in \mathrm{Mod}(S^2, \mathcal{P}'_f)$, we have $\psi(p_0) = 0$, $\psi(p_{n+1}) = 1$, and define $\psi(p_{n+2}) = \infty$, and $x_m := \psi(p_m)$ for $i \in \{1, \ldots, n+1\} - \{j\}$.

Using arguments identical to those in lemma 6.0.2, we construct a Thurston map f', with postcritical set $\mathcal{P}'_f$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Teich}(S^2, \mathcal{P}'_f) \arrow[r, "\sigma'_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}'_f) \arrow[d, "\pi"] \\
\mathbb{P}^{n-1} & \mathbb{P}^{n-1} \arrow[l, "g'"']
\end{tikzcd}$$

Moreover, there is an induced map $g' : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$, which is an endomorphism. Notice that the degree of f' is d^N, and the algebraic degree of g' is also d^N; hence, g' is a postcritically finite enomorphism by corollary 6.0.3. $\square$

Remark 7.3.1. Notice that there is a hyperplane of Δ corresponding to the coalescing of any two of the points in the set $\{p_0, \ldots, p_{n+1}\}$, and note that ∞ is absent. There is no hyperplane in Δ corresponding to the coalescing of ∞ with any of the p_i. This has everything to do with the fact that we are using the $\mathbb{P}^n$ compactification of the moduli space, and for topological polynomials, this is natural (see section 3.2.2).

7.3.1 Completely postcritically finite endomorphisms of $\mathbb{P}^n$

The previous results inspire us to consider the following notion, defined inductively. This definition is a modified version of that found in [21].

Suppose that $G : \mathbb{P}^n \to \mathbb{P}^n$ is holomorphic, and let C_1 be the critical set of G. The set C_1 is algebraic of codimension 1. Define

$$D_1 := \bigcup_{i>0} G^{\circ i}(C_1) \text{ and } E_1 := \bigcap_{i>0} G^{\circ i}(D_1).$$

The set D_1 is precisely the postcritical set of G. Evidently, if D_1 is closed, then E_1 is the ω-limit set of C_1.

Definition 7.3.1. *The map G is* 1-critically finite *if it is postcritically finite (that is, D_1 and hence E_1 are algebraic sets).*

We now define j-critically finite maps of $\mathbb{P}^n$ for $1 < j \leq n$.

Definition 7.3.2. *Suppose that G is $(j-1)$-critically finite. This means, in particular, that the set E_{j-1} has been inductively defined as an algebraic set of codimension $j-1$. Then $C_j := E_{j-1} \cap C_1 = E_{j-1} \cap C_{j-1}$ is algebraic of codimension j. We say that G is* j-critically finite *if $D_j := \cup_{i>0} G^{\circ i}(C_j)$ is algebraic.*

Definition 7.3.3. *Let $G : \mathbb{P}^n \to \mathbb{P}^n$ be a postcritically finite endomorphism. Then G is* completely postcritically finite *if it is n-critically finite.*

Corollary 7.3.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\text{alg deg}\,(g_f) = d$. Then $g_f : \mathbb{P}^n \to \mathbb{P}^n$ is completely postcritically finite.*

Proof. The proof follows immediately from proposition 7.3.1. $\square$

CHAPTER 8

PERIODIC CYCLES

8.1 The semi-group $\Theta_{\mathcal{P}}(R)$

In this section, we define a semi-group which will be essential for the discussion of the periodic cycles of the enomdorphism $g : \mathbb{P}^n \to \mathbb{P}^n$.

Given $\mathcal{P} \subset S^2$, we denote the set of all Thurston maps with postcritical set $\mathcal{P}$ as $\mathrm{Th}_{\mathcal{P}}$. We define an equivalence relation on $\mathrm{Th}_{\mathcal{P}}$ as follows. Let $f, g \in \mathrm{Th}_{\mathcal{P}}$, so we have $\mathcal{P}_f = \mathcal{P}_g = \mathcal{P}$. We say f is *strongly equivalent* to g iff there are homeomorphisms $h_0 : (S^2, \mathcal{P}) \to (S^2, \mathcal{P})$ and $h_1 : (S^2, \mathcal{P}) \to (S^2, \mathcal{P})$ for which $h_0 \circ f = g \circ h_1$ and h_0, h_1 are isotopic to the identity through homeomorphisms agreeing on $\mathcal{P}$. In particular, we have the following commutative diagram:

$$\begin{tikzcd}
(S^2, \mathcal{P}) \arrow[r, "h_1"] \arrow[d, "f"'] & (S^2, \mathcal{P}) \arrow[d, "g"] \\
(S^2, \mathcal{P}) \arrow[r, "h_0"] & (S^2, \mathcal{P}).
\end{tikzcd}$$

If f is strongly equivalent to g, we write $f \sim g$. The relation $\sim$ is an equivalence relation on $\mathrm{Th}_{\mathcal{P}}$, which is finer than Thurston equivalence. Let $R(\Omega, P, \alpha, \nu)$ be a ramification portrait of degree d such that $|\mathcal{P}| = |P|$, and let $\mathrm{Th}_{\mathcal{P}}(R)$ be the set of all Thurston maps realizing R with postcritical set $\mathcal{P}$. Since $\mathrm{Th}_{\mathcal{P}}(R) \subset \mathrm{Th}_{\mathcal{P}}$, the equivalence relation $\sim$ is defined on $\mathrm{Th}_{\mathcal{P}}(R)$. In [23], the author proved that composition is well-defined on the $\sim$-equivalence classes $G_{\mathcal{P}}(R) := \mathrm{Th}_{\mathcal{P}}(R)/\sim$, so $G_{\mathcal{P}}(R)$ generates a semi-group.

Definition 8.1.1. *Let $R(\Omega, P, \alpha, \nu)$ be a ramification portrait of degree d, and let $\mathcal{P} \subset S^2$ be finite such that $|\mathcal{P}| = |P|$. We define $\Theta_{\mathcal{P}}(R) :=< G_{\mathcal{P}}(R) >$, the semi-group generated by $G_{\mathcal{P}}(R)$.*

8.2 Repelling periodic cycles

Theorem 8.2.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d, and identify* $\mathrm{Mod}(S^2, \mathcal{P}_f)$ *with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that* alg deg $(g_f) = d$. *Then the periodic cycles of g_f contained in $\mathbb{P}^n - \Delta$ are repelling.*

Proof. Let $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$, and choose a representative homeomorphism $\phi(S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ normalized in the standard way, Then there exists a unique $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$ normalized in the standard way such that the diagram commutes, where F_ϕ is a polynomial of degree d,

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F_\phi"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f))
\end{tikzcd}$$

and ψ is a representative of $\tau' := \sigma_f(\tau)$.

Since f has the $\pi\sigma$-property, F_ϕ depends only on $\mathbf{x} \in \pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))$. By theorem 6.0.1, $\pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f)) = \mathbb{P}^n - \mathcal{L}$, so F_ϕ depends holomorphically on the points $\mathbf{x} \in \mathbb{P}^n - \mathcal{L}$, and we write $F_\phi = F_{\mathbf{x}}$.

Let $\mathbf{x}_0, \mathbf{x}_2 \dots, \mathbf{x}_{N-1}$ be a periodic cycle of g_f contained in $\mathbb{P}^n - \Delta$. Note that this periodic cycle is necessarily contained in $\mathbb{P}^n - \mathcal{L}$ since the only periodic cycles in $\mathcal{L}$ are contained in Δ.

For each $i \in [0, N-1]$, choose a homeomorphism $\phi_i : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi_i(\mathcal{P}_f))$, so that $\phi_i(p_{n+2}) = 0, \phi_i(p_{n+1}) = 1, \phi_i(\infty) = \infty$, and $\phi_i(p_j) = x_j$ for $j \in [1, n+1]$. For each $\mathbf{x}_i$, consider the polynomial $F_{\mathbf{x}_i}$. Define the family of N Thurston maps as follows

$$f_{\mathbf{x}_i} := \phi^{-1}_{i+1 \pmod N} \circ F_{\mathbf{x}_i} \circ \phi_i.$$

Notice that each $f_{\mathbf{x}_i} \in G_{\mathcal{P}_f}(R_f)$, and we have the following commutative diagram for each i:

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\phi_i"] \arrow[d, "f_{\mathbf{x}_i}"'] & (\mathbb{P}^1, \phi_i(\mathcal{P}_f)) \arrow[d, "F_{\mathbf{x}_i}"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi_{i+1}"] & (\mathbb{P}^1, \phi_{i+1}(\mathcal{P}_f))
\end{tikzcd}$$

and by construction, we have that

$$\sigma_{f_{\mathbf{x}_i}}(\tau_{i+1}) = \tau_i.$$

where τ_i is the element in $\mathrm{Teich}(S^2, \mathcal{P}_f)$ defining the class containing ϕ_i. Moreover, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_{f_{\mathbf{x}_i}}"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"']
\end{tikzcd}$$

which commutes for $i = 0, \ldots, N-1$. We therefore have the following commutative diagram

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\phi_0"] \arrow[d, "f_{\mathbf{x}_{N-1}} \circ \cdots \circ f_{\mathbf{x}_0}"'] & (\mathbb{P}^1, \phi_0(\mathcal{P}_f)) \arrow[d, "F_{\mathbf{x}_{N-1}} \circ \cdots \circ F_{\mathbf{x}_0}"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi_0"] & (\mathbb{P}^1, \phi_0(\mathcal{P}_f))
\end{tikzcd}$$

So the Thurston map $f_{\mathbf{x}_{N-1}} \circ \cdots \circ f_{\mathbf{x}_0}$ is Thurston equivalent to $F_{\mathbf{x}_{N-1}} \circ \cdots \circ F_{\mathbf{x}_0}$, or rather, τ_0 is a fixed point of $\sigma_{f_{\mathbf{x}_{N-1}} \circ \cdots \circ f_{\mathbf{x}_0}}$. Notice also that $\pi(\tau_0) = \mathbf{x}_0$. The composition $F_{\mathbf{x}_{N-1}} \circ \cdots \circ F_{\mathbf{x}_0}$ of polynomials is postcritically finite, whereas the $F_{\mathbf{x}_i}$ themselves are not; the Thurston map $f_{\mathbf{x}_{N-1}} \circ \cdots \circ f_{\mathbf{x}_0} \in \Theta_{\mathcal{P}_f}(R_f)$.

Consider the commutative diagram:

$$\begin{tikzcd}
\operatorname{Teich}(S^2, \mathcal{P}_f) \arrow[rr, "\sigma_{f_{\mathbf{x}_{N-1}} \circ \cdots \circ f_{\mathbf{x}_0}}"] \arrow[d, "\pi"'] & & \operatorname{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & & \mathbb{P}^n \arrow[ll, "g_f^{\circ N}"']
\end{tikzcd}$$

we have found a fixed point of the pullback map: $\sigma_{f_{\mathbf{x}_{N-1}} \circ \cdots \circ f_{\mathbf{x}_0}}(\tau_0) = \tau_0$. Since this map is contracting in the Teichmüller metric, τ_0 is an attracting fixed point, which implies that $\pi(\tau_0)$ must be repelling for $g_f^{\circ N}$ as we can see from the diagram above. $\square$

8.2.1 Fixed points

Notice that the fixed points of g_f in $\mathbb{P}^n - \Delta$ correspond to postcritically finite polynomials; that is, if g_f has a fixed point $\mathbf{z}$, there exists a Thurston map $f' \in G_{\mathcal{P}_f}(R_f)$ such that $\sigma_{f'}$ has a fixed point, τ', and $\pi(\tau') = \mathbf{z}$. In this case, the Thurston map f' is Thurston equivalent to the polynomial $F_{\mathbf{z}}$, which is postcritically finite.

One may naturally inquire about the periodic cycles contained in Δ. It is possible to dynamically classify these periodic cycles using the relevant objects from Thurston's theorem, however, this analysis is much cleaner if we are equipped with the following statement.

Conjecture 8.2.1. *Let $f : S^2 \to S^2$ be a Thurston map of degree d, and identify $\operatorname{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Suppose that there exists an* *endomorphism* $g_f : \mathbb{P}^n \to \mathbb{P}^n$ *making the following diagram commute.*

$$\begin{tikzcd} \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \arrow[r, "\sigma_f"] \arrow[d, "\widetilde{\pi}"'] & \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \arrow[d, "\widetilde{\pi}"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"'] \end{tikzcd}$$

Then the map $\sigma_f : \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \longrightarrow \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)}$ *is surjective.*

We omit the discussion of the periodic cycles of g_f which are contained in Δ, reserving it for a subsequent paper.

CHAPTER 9

EXTENSIONS OF THE MAPS TO DIFFERENT COMPACTIFICATIONS

In chapters 4 and 5, we proved that under some circumstances, the maps

$$g_f : \pi(\sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))) \longrightarrow \mathbb{P}^n - \Delta$$

can be extended to postcritically finite endomorphisms $g_f : \mathbb{P}^n \to \mathbb{P}^n$. In this chapter, we explore the possibility of extending some of these maps to other compactifications discussed in chapter 3. We proceed with the following discussion by analyzing some examples.

9.1 Extending the maps to $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathbb{P}^n}$

As proven in theorem 4.0.1 if f is a Thurston map which is a topological polynomial such that $\Omega_f \subseteq \mathcal{P}_f$, then f has the $\pi\sigma$-property, and there is an induced map $g_f : \mathbb{P}^n \to \mathbb{P}^n$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"']
\end{tikzcd}$$

Moreover, the induced map is a postcritically finite endomophism; in particular, it is holomorphic on $\mathbb{P}^n$.

A natural question to ask if f is a Thurston map such that $\Omega_f \subseteq \mathcal{P}_f$, but f is not a topological polynomial, then

- is there an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ which makes the diagram above commute, and

- if there is an induced map, is it holomorphic?

We provide an example in chapter 10 which gives a negative answer to the first question above, and we now provide an example which gives a negative answer to the second question above.

Example 9.1.1. Let R be the following ramification portrait, which is periodic of degree 2. Note that R is not of polynomial type.

$$\begin{tikzcd} 0 \arrow[r, "2"] & p_1 \arrow[r] & p_2 \arrow[ll, bend left] & & \infty \arrow[r, "2"] & 1 \arrow[l, bend left] \end{tikzcd}$$

Suppose $f : S^2 \to S^2$ is a Thurston map with postcritical set $\mathcal{P}_f = \{0, 1, \infty, p_1, p_2\}$ which realizes R. (Since R is not of polynomial type, we cannot apply theorem 2.4.1, so we might first wonder if such an f exists. Results in [5] imply that indeed such a map does exist: degree 4 is the minimum degree where there is branch data that is not realized). This is the ramification portrait for the *mating of the rabbit and the basilica*.

Let $\tau \in \text{Teich}(S^2, \mathcal{P}_f)$ of which $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a representative homeomorphism, normalized so that $\phi(0) = 0, \phi(1) = 1$, and $\phi(\infty) = \infty$. For notation, suppose $\phi(p_1) = y_1$ and $\phi(p_2) = y_2$. Then $\tau' := \sigma_f(\tau)$ is represented by a homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$, which is normalized so that $\psi(0) = 0, \psi(1) = 1$ and $\psi(\infty) = \infty$, such that the following diagram commutes, where $F_\phi : (\mathbb{P}^1, \psi(\mathcal{P}_f)) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a rational function of degree 2.

$$\begin{tikzcd} (S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F_\phi"] \\ (S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f)) \end{tikzcd}$$

For notation, suppose that $\psi(p_1) = x_1$ and $\psi(p_2) = x_2$. The commutative diagram above implies that

- F_ϕ has two simple critical points at 0 and ∞, and
- $F_\phi(x_2) = 0, F_\phi(\infty) = 1$, and $F_\phi(1) = \infty$, and
- $F_\phi(0) = y_1$, and $F_\phi(x_1) = y_2$.

The first two points above imply that a normal form for F_ϕ is

$$F_\phi(z) = \frac{z^2 - x_2^2}{z^2 - 1}.$$

We see immediately that f has the $\pi\sigma$-property, and that there is an induced map $g_f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$,

$$y_1 = F_\phi(0) = x_2^2, \qquad y_2 = F_\phi(x_1) = \frac{x_1^2 - x_2^2}{x_1^2 - 1}.$$

And in homogeneous coordinates, the map is

$$[x_1 : x_2 : x_3] \longmapsto [x_2^2(x_1^2 - x_3^2) : x_3^2(x_1^2 - x_2^2) : x_3^2(x_1^2 - x_3^2)].$$

This map is not holomorphic on $\mathbb{P}^2$; there are six points of indeterminacy:

$$\mathcal{I} = \{[1:0:0], [0:1:0], [1:1:1], [1:1:-1], [1:-1:1], [-1:1:1]\}.$$

Note that this map has algebraic degree 4, and the topological degree of f is 2. (Compare with corollary 5.2.1).

So if f is not a topological polynomial, the induced map may not be holomorphic on $\mathbb{P}^n$. We may now inquire about the topological polynomials: suppose that f is a Thurston map which is a topological polynomial. Suppose that f has the $\pi\sigma$-property, and that there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Then is $g_f : \mathbb{P}^n \to \mathbb{P}^n$ necessarily holomorphic? The following example provides a negative answer to this question.

Example 9.1.2. Let R be the following ramification portrait of polynomial type of degree 3, which is preperiodic.

$$\alpha \xrightarrow{2} 0 \xrightarrow{2} 1 \longrightarrow p \longrightarrow q \circlearrowright \qquad \infty \circlearrowright 3$$

Suppose $f : S^2 \to S^2$ is a Thurston map with postcritical set $\mathcal{P}_f = \{0, \infty, 1, p, q\}$ which realizes R.

Let $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$ of which $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a representative homeomorphism, normalized so that $\phi(0) = 0, \phi(1) = 1$, and $\phi(\infty) = \infty$. Define $X := \phi(p)$ and $Y = \phi(q)$. Then $\tau' := \sigma_f(\tau)$ is represented by a unique homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$, normalized so that $\psi(0) = 0, \psi(1) = 1$, and $\psi(\infty) = \infty$, such that the following diagram commutes,

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F_\phi"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f))
\end{tikzcd}$$

where $F_\phi : (\mathbb{P}^1, \psi(\mathcal{P}_f)) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a polynomial of degree 3. We define $x := \psi(p), y := \psi(q)$, and $\omega := \psi(\alpha)$. The commutative diagram above implies that

- $F'_\phi(\omega) = 0$ and $F_\phi(\omega) = 0$,
- $F_\phi(0) = 1$ and $F'_\phi(0) = 0$,
- $F_\phi(x) - F_\phi(y) = 0$,
- $F_\phi(1) = X$ and $F_\phi(x) = Y$.

We begin with the following normal form for F_ϕ

$$F_\phi(t) = (t - \omega)^2(At + B),$$

where we have already imposed the first condition above. We impose the second condition to find that $A = 2/\alpha^3$, and $B = 1/\alpha^2$, so

$$F_\phi(t) = \frac{(t-\alpha)^2(2t+\alpha)}{\alpha^3}.$$

We impose the third condition to find that

$$\alpha = \frac{2(x^2+xy+y^2)}{3(x+y)},$$

so

$$F_\phi(t) = \frac{27(x+y)^3}{4(x^2+xy+y^2)^3}t^3 - \frac{27(x+y)^2}{4(x^2+xy+y^2)^2}t^2 + 1,$$

and we see that F_ϕ does have the $\pi\sigma$-property. We find the monic polynomial associated to F_ϕ, which we write as $\widetilde{F_{\mathbf{x}}}$:

$$\widetilde{F_{\mathbf{x}}}(t) = t^3 - \frac{x^2+xy+y^2}{x+y}t^2 + \frac{4(x^2+xy+y^2)^3}{27(x+y)^3.}$$

Note that $\widetilde{F_{\mathbf{x}}}$ induces the map

$$\widetilde{G_f} : (x,y,z) \longmapsto (X,Y,Z)$$

$$X = \frac{(3zx+3zy+x^2+xy+y^2)(3zx+3zy-2x^2-2xy-2y^2)^2}{27(x+y)^3},$$

$$Y = \frac{(x+2y)^2(2x+y)^2(x-y)^2}{27(x+y)^3},$$

$$Z = \frac{4(x^2+xy+y^2)^3}{27(x+y)^3}.$$

which is not holomorphic. This induces a map $g_f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given in homogeneous coordinates as

$$g_f : [x:y:z] \longmapsto [X:Y:Z]$$

$$X = (3zx+3zy+x^2+xy+y^2)(3zx+3zy-2x^2-2xy-2y^2)^2,$$

$$Y = (x+2y)^2(2x+y)^2(x-y)^2,$$

$$Z = 4(x^2+xy+y^2)^3.$$

Note that this map has algebraic degree 6, whereas f has topological degree 3. (Compare with corollary 5.2.1). This map *is not* holomorphic; it has a point of indeterminacy at $\mathcal{I} = \{[0:0:1]\}$.

In conclusion, we see that even if f is a topological polynomial, and there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, this map may not be holomorphic. This example is interesting in light of the fact that the $\mathbb{P}^n$ compactification of the moduli space $\mathrm{Mod}(S^2, \mathcal{P}_f)$ is natural for the topological polynomials, however, the induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ may not be holomorphic (if it exists at all).

9.2 Extending the maps to $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\prod \mathbb{P}^1}$

In this section, we contemplate extending the induced map to $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\prod \mathbb{P}^1}$. We present an example where the induced map extends holomorphically to $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathbb{P}^n}$, but does not extend holomorphically to $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\prod \mathbb{P}^1}$.

Example 9.2.1. Let R be the following periodic ramification portrait of polynomial type, of degree 2.

$$\begin{tikzcd} 0 \arrow[r, "2"] & 1 \arrow[r] & p_1 \arrow[r] & p_2 \arrow[lll, bend left] & & \infty \arrow[loop right, "2"] \end{tikzcd}$$

Suppose the Thurston map $f : S^2 \to S^2$ is topological polynomial of degree 2, with postcritical set $\mathcal{P}_f = \{0, \infty, 1, p_1, p_2\}$, which realizes R.

Let $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$ of which $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a representative homeomorphism, normalized so that $\phi(0) = 0, \phi(1) = 1$ and $\phi(\infty) = \infty$. For notation, suppose $\phi(p_1) = y_1$, and $\phi(p_2) = y_2$. Then $\tau' := \sigma_f(\tau)$ is represented by a unique homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$, normalized so that

$\psi(0) = 0, \psi(1) = 1$, and $\psi(\infty) = \infty$, such that the following diagram commutes, where $F_\phi : (\mathbb{P}^1, \psi(\mathcal{P}_f)) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a polynomial of degree 2.

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F_\phi"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f))
\end{tikzcd}$$

For notation, suppose that $\psi(p_1) = x_1$, and $\psi(p_2) = x_2$. The commutative diagram above implies that

- $F'_\phi(0) = 0, F_\phi(0) = 1$,
- $F_\phi(x_2) = 0$,
- $F_\phi(1) = y_1$, and $F_\phi(x_1) = y_2$.

Imposing the conditions from the first point above implies

$$F_\phi(z) = az^2 + 1,$$

and we can eliminate the parameter a by imposing the condition from the second point gives

$$a = \frac{-1}{x_2^2} \implies F_\phi(z) = 1 - \frac{z^2}{x_2^2}.$$

We obtain the induced map:

$$g_f : (x_1, x_2) \longmapsto \left(1 - \frac{1}{x_2^2}, 1 - \frac{x_1^2}{x_2^2}\right)$$

viewed as a map on $\mathbb{C}^2$. We extend this map to a map on $\mathbb{P}^1 \times \mathbb{P}^1$ in the following way:

$$g_f : ([x_1 : z_1], [x_2 : z_2]) \longmapsto \left([x_2^2 - z_2^2 : x_2^2], [x_2^2 - x_1^2 : x_2^2]\right).$$

Note that this map is not holomorphic on $\mathbb{P}^1 \times \mathbb{P}^1$ as the point $([0 : 1], [0 : 1])$ is a point of indeterminacy: the point $([0 : 1], [0 : 1])$ maps to $([1 : 0], [0 : 0]) \notin \mathbb{P}^1 \times \mathbb{P}^1$.

From theorem 4.0.1, we know that the induced map extension of the induced map $g_f : \mathbb{P}^2 \to \mathbb{P}^2$ is holomorphic.

9.3 Extending the maps to $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}}$

Suppose $f : S^2 \to S^2$ is a Thurston map with postcritical set $\mathcal{P}_f$ for which there is an induced map $g_f : \mathbb{P}^n \to \mathbb{P}^n$. Suppose that the induced map $g_f : \mathbb{P}^n \to \mathbb{P}^n$ is holomorphic. In section 7.2.1 we mention the possibility of a holomorphic lift of g_f to $\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}}$,

$$\widetilde{g_f} : \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}} \longrightarrow \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}}$$

for which the two smaller rectangles of the following diagram commute (the larger rectangle commutes by proposition 7.2.1).

$$\begin{tikzcd}
\overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \arrow[r, "\sigma_f"] \arrow[d, "\overline{\pi}"'] & \overline{\mathrm{Teich}(S^2, \mathcal{P}_f)} \arrow[d, "\overline{\pi}"] \\
\overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}} \arrow[d, "\beta"'] & \overline{\mathrm{Mod}(S^2, \mathcal{P}_f)}_{\mathrm{DM}} \arrow[l, "\widetilde{g}_f"'] \arrow[d, "\beta"] \\
\mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"']
\end{tikzcd}$$

In this section, we manufacture a Thurston map f for which there is an induced map $g_f : \mathbb{P}^n \to \mathbb{P}^n$ which is a postcritically finite endomorphism, but we prove that there is no such lift.

Example 9.3.1. We begin with the same set-up as in example 9.2.1. Let R be the following periodic ramification portrait of polynomial type, of degree 2.

$$\begin{tikzcd} 0 \arrow[r, "2"] & 1 \arrow[r] & p_1 \arrow[r] & p_2 \arrow[lll, bend left] \end{tikzcd} \qquad \infty \circlearrowright 2$$

Suppose the Thurston map $f : S^2 \to S^2$ is topological polynomial of degree 2, with postcritical set $\mathcal{P} = \{0, \infty, 1, p_1, p_2\}$, which realizes R.

Let $\tau \in \text{Teich}(S^2, \mathcal{P})$ of which $\phi : (S^2, \mathcal{P}) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}))$ is a representative homeomorphism, normalized so that $\phi(0) = 0, \phi(1) = 1$ and $\phi(\infty) = \infty$. For notation, suppose $\phi(p_1) = y_1$, and $\phi(p_2) = y_2$. Then $\tau' := \sigma_f(\tau)$ is represented by a unique homeomorphism $\psi : (S^2, \mathcal{P}) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}))$, where we normalize so that $\psi(0) = 0, \psi(1) = 1$, and $\psi(\infty) = \infty$, such that the following diagram commutes, where $F_\phi : (\mathbb{P}^1, \psi(\mathcal{P})) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}))$ is a polynomial of degree 2.

$$\begin{tikzcd} (S^2, \mathcal{P}) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P})) \arrow[d, "F_\phi"] \\ (S^2, \mathcal{P}) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P})) \end{tikzcd}$$

For notation, suppose that $\psi(p_1) = x_1$, and $\psi(p_2) = x_2$. The commutative diagram above implies that

- $F'_\phi(0) = 0, F_\phi(0) = 1$,
- $F_\phi(x_2) = 0$,
- $F_\phi(1) = y_1$, and $F_\phi(x_1) = y_2$.

Imposing the conditions from the first point above implies

$$F_\phi(z) = az^2 + 1,$$

and we can eliminate the parameter a by imposing the condition from the second point gives

$$a = \frac{-1}{x_2^2} \implies F_\phi(z) = 1 - \frac{z^2}{x_2^2}.$$

We obtain the induced map $g_f : \mathbb{P}^2 \to \mathbb{P}^2$:

$$g_f : [x_1 : x_2 : x_3] \longmapsto [x_2^2 - x_3^2 : x_2^2 - x_1^2 : x_2^2].$$

From theorem 4.0.1, the extension of the induced map $g_f : \mathbb{P}^2 \to \mathbb{P}^2$ is holomorphic, and we know from theorem 4.0.1 that this is a postcritically finite endomorphism, and for any Thurston map f which realizes R, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Teich}(S^2, \mathcal{P}) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}) \arrow[d, "\pi"] \\
\mathbb{P}^2 & \mathbb{P}^2 \arrow[l, "g_f"']
\end{tikzcd}$$

We now contemplate the existence of a lift

$$\widetilde{g_f} : \overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}} \longrightarrow \overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}},$$

such that the following diagram commutes.

$$\begin{tikzcd}
\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}} \arrow[r, "\widetilde{g_f}"] \arrow[d, "\beta"'] & \overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}} \arrow[d, "\beta"] \\
\mathbb{P}^2 \arrow[r, "g_f"] & \mathbb{P}^2
\end{tikzcd}$$

Recall from section 3.3.2 that in the case where $|\mathcal{P}| = 5$, then $\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}}$ is isomorphic to $\mathbb{P}^2$ blown up at the four triple intersection points in the figure below. Since $g_f : \mathbb{P}^2 \to \mathbb{P}^2$ is holomorphic, this map necessarily lifts as follows;

$$\begin{tikzcd}
\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}} \arrow[d, "\beta"'] \arrow[dr, "\widetilde{g_f}"] & \\
\mathbb{P}^2 \arrow[r, "g_f"] & \mathbb{P}^2
\end{tikzcd}$$

that is, g_f extends holomorphically to the exceptional divisors at each of the four triple points. The real question is, does the map $g_f : \mathbb{P}^2 \to \mathbb{P}^2$ lift to a dynamical

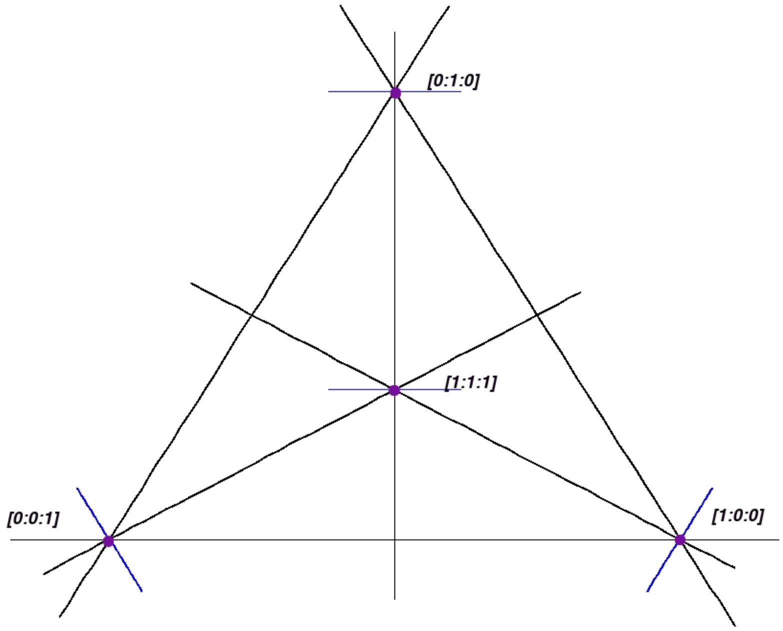


Figure 9.1: We obtain $\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}}$ for $|\mathcal{P}| = 5$ by blowing up $\mathbb{P}^2$ at the four points of triple intersection. The black lines are the elements of Δ, and the blue lines are the exceptional divisors.

system? That is, does it lift so that the following diagram commutes?

$$\begin{tikzcd}
\overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}} \arrow[r, "\widetilde{g}_f"] \arrow[d, "\beta"'] & \overline{\mathrm{Mod}(S^2, \mathcal{P})}_{\mathrm{DM}} \arrow[d, "\beta"] \\
\mathbb{P}^2 \arrow[r, "g_f"] & \mathbb{P}^2
\end{tikzcd}$$

This is not promising as

$$g_f^{-1}\left(\{[1:1:1],[0:0:1],[0:1:0],[1:0:0]\}\right) \not\subset$$

$$\{[1:1:1],[0:0:1],[0:1:0],[1:0:0]\}\,,$$

and in particular,

$$g_f^{-1}([0:0:1]) = \{[1:1:1],[-1:1:1],[1:-1:1],[1:1:-1]\}.$$

Let $E_{[0:0:1]}$ denote the exceptional divisor at the point $[0:0:1]$. Then in $\overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}}$, the map $\widetilde{g_f}$ has new points of indeterminacy at the points $[-1:1:1],[1:-1:1]$, and $[1:1:-1]$ as these points map to $E_{[0:0:1]}$. To resolve these points of indeterminacy, we should blow up each of these three inverse images of $[0:0:1]$, however, we are not entitled to do such a thing as $\overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}}$ is isomorphic to $\mathbb{P}^2$ blown up *only* at the four triple points in the figure above. Notice that $[1:1:1]$ is not a point of indeterminacy of $\widetilde{g_f}$ as this point has already been blown up to obtain $\overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}}$. So for this particular map $g_f : \mathbb{P}^2 \to \mathbb{P}^2$, there is no lift

$$\widetilde{g_f} : \overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}} \longrightarrow \overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}},$$

such that the following diagram commutes.

$$\begin{tikzcd}
\overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}} \arrow[r, "\widetilde{g_f}"] \arrow[d, "\beta"'] & \overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}} \arrow[d, "\beta"] \\
\mathbb{P}^2 \arrow[r, "g_f"] & \mathbb{P}^2
\end{tikzcd}$$

We now contemplate whether the following diagram commutes for all Thurston maps $f : S^2 \to S^2$ which realize portrait R.

$$\begin{tikzcd}
\overline{\mathrm{Teich}(S^2,\mathcal{P})} \arrow[r, "\sigma_f"] \arrow[d, "\bar{\pi}"'] & \overline{\mathrm{Teich}(S^2,\mathcal{P})} \arrow[d, "\bar{\pi}"] \\
\overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}} & \overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}} \arrow[l, "\widetilde{g}_f"']
\end{tikzcd}$$

By definition, the above diagram commutes if

$$\widetilde{g}_f\left(\bar{\pi}\left(\sigma_f\left(\tau\right)\right)\right) = \bar{\pi}(\tau)$$

for all $\tau \in \overline{\mathrm{Teich}(S^2,\mathcal{P})}$.

Lemma 9.3.1. *The diagram above commutes if and only if*

$$\{[-1:1:1],[1:-1:1],[1:1:-1]\} \cap \bar{\pi}\left(\sigma_f\left(\overline{\mathrm{Teich}(S^2,\mathcal{P})}\right)\right) = \emptyset.$$

Proof. This follows immediately from the discussion above. □

Proposition 9.3.1. *There exists a Thurston map $f : S^2 \to S^2$ which realizes R, such that* $[-1:1:1] \in \bar{\pi}\left(\sigma_f\left(\overline{\mathrm{Teich}(S^2,\mathcal{P}_f)}\right)\right)$.

Proof. Define the closed sets

$$A := \bar{\pi}^{-1}\left(\{[1:1:1],[-1:1:1],[1:-1:1],[1:1:-1]\}\right) \subset \overline{\mathrm{Teich}(S^2,\mathcal{P}_f)},$$

and

$$B := \bar{\pi}^{-1}\left([-1:1:1]\right) \subset A.$$

Notice that

$$\sigma_f\left(\bar{\pi}^{-1}\left(E_{[1:0:0]}\right)\right) \subseteq A.$$

Fix $b \in B$, and define

$$\epsilon := d(b, A - B)_{WP}.$$

Choose $\tau' \in \text{Teich}(S^2, \mathcal{P}_f)$ so that $d(\tau', b) < \epsilon/2$.

Let $c \in \bar{\pi}^{-1}\left(E_{[1:0:0]}\right) \subset \overline{\text{Teich}(S^2, \mathcal{P}_f)}$, and choose $\tau \in \text{Teich}(S^2, \mathcal{P}_f)$ such that

$$d(\tau, c)_{WP} < \frac{\epsilon}{2\sqrt{2}}.$$

We now manufacture a Thurston map $f : S^2 \to S^2$ which realizes portrait R, such that $\sigma_f(\tau) = \tau'$. Recall the polynomial

$$F_{\mathbf{x}}(z) = 1 - \frac{z^2}{x_2^2}.$$

which induced the map $g_f : \mathbb{P}^2 \to \mathbb{P}^2$. Set $\mathbf{x} := \pi(\tau')$, and we have such a polynomial $F_{\mathbf{x}}$. Choose a homeomorphism $\phi : (S^2, \mathcal{P}) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}))$, in the class of homeomorphisms defined by τ, and normalize so that $\phi(0) = 0, \phi(1) = 1$, and $\phi(\infty) = \infty$. Choose a homeomorphism $\psi : (S^2, \mathcal{P}) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}))$, in the class of homeomorphisms defined by τ', and normalize so that $\psi(0) = 0, \psi(1) = 1$, and $\psi(\infty) = \infty$. Define the Thurston map

$$f := \phi^{-1} \circ F_{\mathbf{x}} \circ \psi.$$

This Thurston map has postcritical set $\mathcal{P}$, and it realizes R. Moreover, $\sigma_f(\tau) = \tau'$, by construction.

By theorem 7.2.2,

$$\sigma_f : \overline{\text{Teich}(S^2, \mathcal{P})} \longrightarrow \overline{\text{Teich}(S^2, \mathcal{P})}$$

is $\sqrt{2}$-Lipschitz. We have

$$d(\sigma_f(\tau), \sigma_f(c))_{WP} \leq \sqrt{2} d(\tau, c)_{WP} < \sqrt{2}\left(\frac{\epsilon}{2\sqrt{2}}\right) = \epsilon/2.$$

and so

$$d(\tau', \sigma_f(c)) < \epsilon/2.$$

If the diagram is to commute, then we must have $\sigma_f(c) \subset A$, but from the estimates above, we must have $\sigma_f(c) \in B$, which implies that

$$[-1:1:1] \in \bar{\pi}\left(\sigma_f\left(\overline{\mathrm{Teich}(S^2,\mathcal{P})}\right)\right).$$

Hence, by lemma 9.3.1, the diagram

$$\begin{array}{ccc} \overline{\mathrm{Teich}(S^2,\mathcal{P})} & \xrightarrow{\sigma_f} & \overline{\mathrm{Teich}(S^2,\mathcal{P})} \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \bar{\pi}} \\ \overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}} & \xleftarrow{\widetilde{g}_f} & \overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}} \end{array}$$

does not commute. $\square$

The arguments outlined in the above example should carry over to higher dimensions; that is, one could show for many more examples that if the induced map extends to $\mathbb{P}^n$, then it does not extend to $\overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}}$. There was nothing special about the fact that $|\mathcal{P}| = 5$, other than the fact that $\overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}}$ becomes more complicated for $|\mathcal{P}| > 5$. However, an analogous argument could be made, just using more complicated combinatorics to understand $\overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathrm{DM}}$.

The example 9.2.1 is quite interesting. Of the three compactifications, the $\overline{\mathrm{Mod}(S^2,\mathcal{P})}_{\mathbb{P}^n}$ "works the best" in the sense that the induced map $g_f : \mathbb{P}^2 \to \mathbb{P}^2$ extends holomorphically. Note that in this example, R is of polynomial type. This is consistent with the remarks made in chapter 3. Thus, when working with Thurston maps which are topological polynomials, it is more natural to consider the $\mathbb{P}^n$ compactification of moduli space.

CHAPTER 10

STATIC PORTRAITS AND MINIMAL PORTRAITS

Let $f : S^2 \to S^2$ be a Thurston map of degree d with postcritical set $\mathcal{P}_f$. One can naturally inquire

- is there a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ making the diagram

$$\begin{tikzcd} \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, dashed, "g_f"'] \end{tikzcd}$$

  commute? And

- can the map g_f be extended to a holomorphic map $g_f : \mathbb{P}^n \to \mathbb{P}^n$?

In this chapter we discuss these questions in detail and introduce *static portraits* and *minimal portraits*, two tools that are remarkably simple but very useful in analyzing the questions above. We motivate the discussion that follows with an example.

As proven in theorem 4.0.1, if R is a periodic ramification portrait of polynomial type, of degree d, and if f is any Thurston map realizing R, then there exists a postcritically finite endomorphism $g_f : \mathbb{P}^n \to \mathbb{P}^n$ such that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathrm{Teich}(S^2, \mathcal{P}_f) \arrow[d, "\pi"] \\ \mathbb{P}^n & \mathbb{P}^n \arrow[l, "g_f"'] \end{tikzcd}$$

It is quite natural to wonder if this result generalizes further; for example, if R is a periodic ramification portrait of degree d, which is not of polynomial type, then is there a map $g_f : \mathbb{P}^n \to \mathbb{P}^n$, which makes the diagram above commute? The following example provides a negative answer.

Example 10.0.2. Let R be the following ramification portrait, which is periodic of degree 3. Note that R is not of polynomial type.

$$\begin{tikzcd} 0 \arrow[loop right, "2"] & 1 \arrow[loop right, "2"] & \infty \arrow[loop right, "2"] & p \arrow[loop right, "2"] \end{tikzcd}$$

Suppose $f : S^2 \to S^2$ is a Thurston map with postcritical set $\mathcal{P}_f = \{0, 1, \infty, p\}$ which realizes R. (Since R is not of polynomial type, we cannot apply theorem 2.4.1, so we might first wonder if such an f exists. Results in [5] imply that it indeed does: degree 4 is the minimum degree where branch data is not realized).

Let $\tau \in \text{Teich}(S^2, \mathcal{P}_f)$ of which $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a representative homeomorphism, normalized so that $\phi(0) = 0, \phi(1) = 1$, and $\phi(\infty) = \infty$. For notation, suppose $\phi(p) = X$. Then $\tau' := \sigma_f(\tau)$ is represented by a homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$, so that $\psi(0) = 0, \psi(1) = 1$ and $\psi(\infty) = \infty$, such that the following diagram commutes,

$$\begin{tikzcd} (S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F_\phi"] \\ (S^2, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f)) \end{tikzcd}$$

where $F_\phi : (\mathbb{P}^1, \psi(\mathcal{P}_f)) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a rational function of degree 3. For notation, suppose that $\psi(p) = x$. The commutative diagram above implies that

- F_ϕ has four simple critical points at: $0, 1, \infty$, and x, and
- $F_\phi(0) = 0, F_\phi(1) = 1, F_\phi(\infty) = \infty$, and $F_\phi(x) = X$.

Imposing the conditions that $F_\phi(\infty) = \infty, F_\phi(0) = 0, F_\phi(1) = 1, F'_\phi(0) = 0$, and $F'_\phi(\infty) = 0$ implies that a normal form for F_ϕ is

$$F_\phi(z) = \frac{z^2(az + b)}{z + a + b - 1}$$

where a and b are complex parameters. Imposing the condition that $F'_\phi(1) = 0$, we have

$$a = \frac{1-2b}{3},$$

and so $F_\phi(z)$ becomes

$$F_\phi(z) = \frac{z^2(z - 2bz + 3b)}{3z + b - 2}.$$

We have determined F_ϕ up to the parameter b. If f has the $\pi\sigma$-property, then we should be able to express the parameter b in terms of x. To this end, we consider our remaining two equations: $F'_\phi(x) = 0$ and $F_\phi(x) = X$ which imply

$$b^2 + 2(x-1)b - x = 0 \text{ and } x^2(x - 2bx + 3b) - X(3x + b - 2) = 0.$$

We eliminate the parameter b from these last two equations obtaining:

$$b = \frac{x^3 - 3Xx - 2X}{2x^3 - 3x^2 + X} \text{ and } x^4 - 4x^3X + 6x^2X - 4xX + X^2 = 0.$$

So we see that the parameter b cannot be expressed in terms of x, but in terms of both x and X. Notice that the final equation

$$x^4 - 4x^3X + 6x^2X - 4xX + X^2 = 0$$

defines a correspondence on $\mathbb{P}^1 \times \mathbb{P}^1$, which is degree 4 in x and degree 2 in X. Thus, there is no map $g_f : \mathbb{P}^1 \to \mathbb{P}^1$, which makes the diagram commute. Note that in the cases where there is such a map g_f, the degree in the variable X would be 1.

The above example indicates that there was something special about the polynomial portraits as opposed to portraits for general Thurston maps. However, it is certainly not the case that a map g_f exists for any ramification portrait of polynomial type; in fact, example 5.2.1 contains such a portrait.

In proposition 5.1.4, we saw that for a general Thurston map, one does not obtain a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, but rather, one obtains a *correspondence*.

We now introduce static portraits; these objects are essential for the discussion that follows.

10.1 Static portraits

Inspired by the calculations in chapters 4 and 5, we define *static portraits* . These objects are combinatorial in nature, but while ramification portraits are dynamical, static portraits are not.

Definition 10.1.1. *We say that* $\mathrm{St}(A, B, \alpha, \nu)$ *is a* static portrait of degree d *if* A *and* B *are finite sets, and there are maps* $\alpha : A \to B$ *and* $\nu : A \to \mathbb{N}$ *such that*

- $\sum_{a\in A}(\nu(a) - 1) = 2d - 2$, *and*
- $\forall\, b \in B$, $\sum_{a\in\alpha^{-1}(b)} \nu(a) \le d$.

Just as for ramification portraits, we define what it means for two static portraits to be isomorphic.

Definition 10.1.2. *Let* $\mathrm{St}_1 := \mathrm{St}(A_1, B_1, \alpha_1, \nu_1)$ *and* $\mathrm{St}_2 := \mathrm{St}(A_2, B_2, \alpha_2, \nu_2)$ *be static portraits. Then* St_1 *and* St_2 *are* isomorphic *if there are bijections* $\beta : A_1 \to A_2$ *and* $\delta : B_1 \to B_2$ *such that the following two diagrams commute,*

$$\begin{tikzcd} A_1 \arrow[r, "\beta"] \arrow[d, "\alpha_1"'] & A_2 \arrow[d, "\alpha_2"] \\ B_1 \arrow[r, "\delta"] & B_2 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[rr, "\beta"] \arrow[dr, "\nu_1"'] & & A_2 \arrow[dl, "\nu_2"] \\ & \mathbb{N} & \end{tikzcd}$$

and we write $\mathrm{St}_1 \sim_{iso} \mathrm{St}_2$. *Any a pair of maps* (β, δ) *is called an* isomorphism *between* St_1 *and* St_2.

Each ramification portrait $R(\Omega, P, \alpha, \nu)$ of degree d, is naturally a static portrait of degree d: $\mathrm{St}(R) := \mathrm{St}(\Omega \cup P, P, \alpha, \nu)$.

Example 10.1.1. If $f(z) = z^2 + i$, then R_f is represented by:

$$\begin{tikzcd} 0 \arrow[r, "2"] & i \arrow[r] & -1+i \arrow[r] & -i \arrow[l, bend left] & & \infty \arrow[loop right, "2"] \end{tikzcd}$$

and $\mathrm{St}(R_f)$ is represented by

$$\begin{tikzcd} 0 \arrow[r, "2"] & i' & i \arrow[r] & (-1+i)' & -1+i \arrow[r] & (-i)' & \infty \arrow[r, "2"] & \infty' \\ & & & -i \arrow[u] & & & & \end{tikzcd}$$

Notice that the domain and range are not identified, so the domain elements are not identified with the range elements. The elements in the range are denoted with a " $'$ ". The arrows above represent the action of the map α, and the numbers above the arrows represent the map ν, where there is no number over the arrow from $x \mapsto y$ iff $\nu(x) = 1$.

10.1.1 Thurston maps

Let $f : S^2 \to S^2$ be a Thurston map of degree d with postcritical set $|\mathcal{P}_f|$. Identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way.

Let $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$ of which $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a representative homeomorphism, normalized in the standard way. Then there is a unique homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$, normalized in the standard way, such that the diagram commutes, where F_ϕ is a rational function of degree d,

$$\begin{tikzcd} (S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & \big(\mathbb{P}^1, \psi(\mathcal{P}_f)\big) \arrow[d, "F_\phi"] \\ (S^2, \mathcal{P}_f) \arrow[r, "\phi"] & \big(\mathbb{P}^1, \phi(\mathcal{P}_f)\big). \end{tikzcd}$$

where ψ represents $\tau' := \sigma_f(\tau)$.

Observe that

$$F_\phi|_{\Omega_{F_\phi}\cup\psi(\mathcal{P}_f)} : \Omega_{F_\phi} \cup \psi(\mathcal{P}_f) \longrightarrow \phi(\mathcal{P}_f),$$

where Ω_{F_ϕ} is the set of critical points of the map F_ϕ. We can therefore consider the static portrait $\mathrm{St}(\Omega_{F_\phi} \cup \psi(\mathcal{P}_f), \phi(\mathcal{P}_f), F_\phi, \text{loc deg } F_\phi)$.

Definition 10.1.3. *Let F_ϕ be the map defined above. The* static portrait associated to the map F_ϕ *is* $\mathrm{St}(F_\phi) := \mathrm{St}(\Omega_{F_\phi} \cup \psi(\mathcal{P}_f), \phi(\mathcal{P}_f), F_\phi, \text{loc deg } F_\phi)$.

Since the diagram above commutes, it is clear that $\mathrm{St}(F_\phi) \sim_{\text{iso}} \mathrm{St}(R_f)$.

10.1.2 Equations and correspondences

Let $f : S^2 \to S^2$ be a Thurston map of degree d, and identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way. Recall the map

$$\Pi : \mathrm{Teich}(S^2, \mathcal{P}_f) \times \sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f)) \longrightarrow \mathrm{Mod}(S^2, \mathcal{P}_f) \times \mathrm{Mod}(S^2, \mathcal{P}_f)$$

given by

$$\Pi : (\tau, \sigma_f(\tau)) \longmapsto (\pi(\tau), \pi(\sigma_f(\tau))).$$

By proposition 5.1.4,

$$V_f := \Pi\left(\mathrm{Teich}(S^2, \mathcal{P}_f) \times \sigma_f(\mathrm{Teich}(S^2, \mathcal{P}_f))\right)$$

is an algebraic subvariety of $\mathbb{P}^n \times \mathbb{P}^n$.This algebraic subvariety defines a correspondence in $\mathbb{P}^n \times \mathbb{P}^n$. In proposition 5.1.5, we saw that the degree of $\rho_2(V_f)$ is equal to 1, if and only it there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$.

We now discuss this correspondence in terms of static portraits, specifically for topological polynomials. The discussion for general Thurston maps is analogous.

Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d with postcritical set $|\mathcal{P}_f|$. Identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way.

Let $\tau \in \mathrm{Teich}(S^2, \mathcal{P}_f)$ of which $\phi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \phi(\mathcal{P}_f))$ is a representative homeomorphism, normalized in the standard way. Then there is a unique homeomorphism $\psi : (S^2, \mathcal{P}_f) \longrightarrow (\mathbb{P}^1, \psi(\mathcal{P}_f))$, normalized in the standard way, such that the diagram commutes, where F_ϕ is a rational function of degree d,

$$\begin{tikzcd}
(S^2, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & \big(\mathbb{P}^1, \psi(\mathcal{P}_f)\big) \arrow[d, "F_\phi"] \\
(S^2, \mathcal{P}_f) \arrow[r, "\phi"] & \big(\mathbb{P}^1, \phi(\mathcal{P}_f)\big).
\end{tikzcd}$$

where ψ represents $\tau' := \sigma_f(\tau)$.

The map F_ϕ is a polynomial of degree d, and so it is of the form

$$F_\phi(z) = a_d z^d + \ldots + a_i z^i + \ldots a_0,$$

where the a_l are complex numbers determined by f and ϕ. Consider the static portrait $\mathrm{St}(\Omega_{F_\phi} \cup \psi(\mathcal{P}_f), \phi(\mathcal{P}_f), F, \mathrm{loc\,deg}\, F_\phi)$. For each $w_i \in \Omega_{F_\phi} \cup \psi(\mathcal{P}_f)$, there is a set of equations Eq_{w_i}. The point w_i satisfies equations of the form

$$F_\phi(w_i) = y_j, \text{ for some } y_j \in \phi(\mathcal{P}_f).$$

But w_i may also satisfy one or both types of the following equations:

1. $F_\phi^{(m)}(w_i) = 0$ where $F_\phi^{(m)}$ is the mth derivative of F_ϕ,
2. $F_\phi(w_i) - F_\phi(t) = 0, \text{ for some } t \in \Omega_{F_\phi} \cup \psi(\mathcal{P}_f)$.

Define the set of equations associated to the static portrait $\mathrm{St}(F_\phi)$ to be

$$Eq_{F_\phi} := \bigcup_{w_i \in \Omega_{F_\phi} \cup \psi(\mathcal{P}_f)} Eq_{w_i}.$$

By construction, Eq_{F_ϕ} is composed entirely of algebraic equations involving the a_l, and the variables x_i, y_j, and α_k, where

$$x_i \in \psi(\mathcal{P}_f), y_j \in \phi(\mathcal{P}_f), \text{ and } \alpha_k \in \Omega_{F_\phi} - \Omega_{F_\phi} \cap \psi(\mathcal{P}_f).$$

We can eliminate the α_k and the a_l from all of the equations to obtain a new set of equations involving only the variables $x_i \in \psi(\mathcal{P}_f)$ and $y_j \in \phi(\mathcal{P}_f)$. Notice that the x_i and y_j are the *moduli space variables*, whereas the α_k were not. The new set of equations in the x_i and y_j contain the correspondence defining the algebraic subvariety V_f. Note that we can solve for each of the y_j in terms of the x_i if and only if there is an induced map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$.

To determine if f has the $\pi\sigma$-property, we need to write each of the a_l as a function of the points in $\psi(\mathcal{P}_f)$, not the points in $\phi(\mathcal{P}_f)$; hence we restrict our attention to the *minimal portrait associated to* F_ϕ.

Consider the set

$$D_{F_\phi} := \Omega_{F_\phi} \cup \{w \in \Omega_{F_\phi} \cup \psi(\mathcal{P}_f) \ : \ \exists\, w' \neq w \text{ where } F_\phi(w) = F_\phi(w')\}.$$

Definition 10.1.4. *Let F_ϕ be the map defined above. The* minimal portrait associated to the map F_ϕ *is*

$$\mathrm{Min}(F_\phi) := \mathrm{St}\left(D_{F_\phi} \cap \psi(\mathcal{P}_f), F_\phi(D_{F_\phi} \cap \psi(\mathcal{P}_f)), F_\phi, \text{loc deg } F_\phi\right).$$

We revisit example 10.1.1.

Example 10.1.2. If $f(z) = z^2 + i$, then R_f is represented by:

$$\begin{tikzcd} 0 \arrow[r, "2"] & i \arrow[r] & -1+i \arrow[r] & -i \arrow[l, bend left] & & \infty \arrow[loop right, "2"] \end{tikzcd}$$

and $\mathrm{Min}(F_\phi)$ is represented by

$$\begin{tikzcd} 0 \arrow[r, "2"] & i' & i \arrow[r] & (-1+i)' & \infty \arrow[r, "2"] & \infty' \\ & & & -i \arrow[u] & & \end{tikzcd}$$

Consider the subset $eq_{F_\phi} \subseteq Eq_{F_\phi}$ which consists of all equations in Eq_{F_ϕ} except those which involve the y_j, (the marked points in the range of F_ϕ). The equations in the set eq_{F_ϕ} involve only points in $\Omega_{F_\phi} \cup \psi(\mathcal{P}_f)$. Each equation in eq_{F_ϕ} is of the form

$$F_\phi^{(m)}(w_i) = 0 \text{ where } F_\phi^{(m)} \text{ is the } m\text{th derivative of } F_\phi, \text{ or}$$

$$F_\phi(w_i) - F_\phi(t) = 0, \text{ for some } t \in \Omega_{F_\phi} \cup \psi(\mathcal{P}_f)$$

for some $w_i \in \Omega_{F_\phi} \cup \psi(\mathcal{P}_f)$. That is, each equation in eq_{F_ϕ} involves only the a_l as well as the variables $x_i \in \psi(\mathcal{P}_f)$, and $\alpha_k \in \Omega_{F_\phi} - \Omega_{F_\phi} \cap \psi(\mathcal{P}_f)$. The cardinality of D_{F_ϕ} is precisely the number of variables x_i and α_k in the equations of eq_{F_ϕ}.

Define the set of equations associated to $\mathrm{Min}(F_\phi)$ to be eq_{F_ϕ}.

Definition 10.1.5. *The number $|D_{F_\phi} \cap \psi(\mathcal{P}_f)|$ is called the* rank *of R_f.*

We will return to this definition later in the chapter.

We can eliminate all of the α_k from the equations in eq_{F_ϕ}, and if we can then solve for each of the a_l uniquely in terms of the x_i, then f clearly has the $\pi\sigma$-property. Otherwise, if the solution is not unique, or if we cannot solve for the a_l, then V_f is a correspondence. However, if V_f is reducible, f may still have the $\pi\sigma$-property, and there could still be an induced map. This happens in the unicritical case (see [23]).

Definition 10.1.6. *Let F_ϕ, a_l and eq_{F_ϕ} be as above. We say that eq_{F_ϕ} is* solvable *if the equations of eq_{F_ϕ} can be solved uniquely for the a_l.*

The following lemma asserts that if $\mathrm{St}(F_{\phi_1})$ is isomorphic to $\mathrm{St}(F_{\phi_2})$, then the sets of equations $Eq_{F_{\phi_1}}$ and $Eq_{F_{\phi_2}}$ are the same (up to a change of variables).

Lemma 10.1.1. *Let the Thurston maps f_1 and f_2 be topological polynomials of degree d such that $\mathrm{St}(F_{\phi_1})$ is isomorphic to $\mathrm{St}(F_{\phi_2})$. Then the equations contained in $Eq_{F_{\phi_1}}$ are the same as the equations contained in $Eq(F_{\phi_2})$, up to a change of variables.*

Proof. The follows immediately from the definitions. $\square$

The above lemma implies that the equations of eq_{ϕ_1} and the equations of $eq_{F_{\phi_2}}$ are the same up to a change of variables.

Proposition 10.1.1. *Let f_1 and f_2 be Thurston maps with postcritical sets $\mathcal{P}_{f_1}$ and $\mathcal{P}_{f_2}$ respectively. Choose some normalization, identifying $\mathrm{Mod}(S^2, \mathcal{P}_{f_1})$ with $\mathbb{P}^{n_1} - \Delta_1$, and $\mathrm{Mod}(S^2, \mathcal{P}_{f_2})$ with $\mathbb{P}^{n_2} - \Delta_2$. Suppose that $\mathrm{Min}(F_{\phi_1})$ is isomorphic to $\mathrm{Min}(F_{\phi_2})$. Then*

- *$eq_{F_1} := eq_{F_{\phi_1}}$ is solvable if and only if $eq_{F_2} := eq_{F_{\phi_2}}$ is solvable,*
- *there is an induced map $g_{f_1} : \mathbb{P}^{n_1} \dashrightarrow \mathbb{P}^{n_1}$ if and only if there is an induced map $g_{f_2} : \mathbb{P}^{n_2} \dashrightarrow \mathbb{P}^{n_2}$, and*

- alg deg $(g_{f_1}) =$ alg deg (g_{f_2})

Proof. The proof of the first point above follows directly from lemma 10.1.1. We now proceed with the proof of the second part. To prove the second point above, we require some notation. Enumerate the elements of $D_{F_{\phi_1}} \cap \psi_1(\mathcal{P}_{f_1})$ as $x_1, \ldots, x_N$, and enumerate the elements of $D_{\phi_{F_2}} \cap \psi_2(\mathcal{P}_{f_2})$ as $y_1, \ldots, y_N$,. Enumerate the elements of $F_{\phi_1}(D_{F_{\phi_1}} \cap \psi_1(\mathcal{P}_{f_1}))$ as $X_1, \ldots X_M$, and enumerate the elements of $F_{\phi_2}(D_{\phi_{F_2}} \cap \psi_2(\mathcal{P}_{f_2}))$ as $Y_1, \ldots, Y_M$,. Since $\mathrm{Min}(F_{\phi_1})$ is isomorphic to $\mathrm{Min}(F_{\phi_2})$, there is an isomorphism (β, δ). We may assume that we have enumerated the x_i and y_i and the X_i and Y_i in a way which reflects this isomorphism, that is, suppose that $\beta(x_i) = y_i$ and suppose that $\delta(X_i) = Y_i$. Suppose there is an induced map $g_{f_1} : \mathbb{P}^{n_1} \dashrightarrow \mathbb{P}^{n_1}$. Then for each $i \in [1, M]$, we can write

$$X_i := X_i(x_1, \ldots, x_N),$$

which means for each $i \in [1, M]$, we have

$$Y_i := Y_i(y_1, \ldots, y_N).$$

Since there is an induced map $g_{f_1} : \mathbb{P}^{n_1} \dashrightarrow \mathbb{P}^{n_1}$, f_1 has the $\pi\sigma$-property, and hence f_2 does as well. Consider the remaining data in the static portrait of F_{ϕ_2}. Recall that

$$F_{\phi_2} : \Omega_{F_{\phi_2}} \cup \psi(\mathcal{P}_f) \longrightarrow \phi(\mathcal{P}_f)$$

is surjective. Recall that in order for F_{ϕ_2} to induce a map, we must write each $Y \in \phi(\mathcal{P}_f)$ as a function of the points of $\psi(\mathcal{P}_f)$. Thus far, we have written $Y_1, \ldots, Y_M$ as functions of the $y_1, \ldots, y_N$. We must now take care of the remaining points of $\phi(\mathcal{P}_f)$. Let $Y \in \phi(\mathcal{P}_f)$. Then

1. Y is not a critical value of F_{ϕ_2}, and

2. there exists a unique $y \in \psi(\mathcal{P}_f)$ so that $F_{\phi_2}(y) = Y$.

For otherwise, Y would have been in $F_{\phi_2}(D_{F_{\phi_2}})$. Thus, $Y = F_{\phi_2}(y)$, which is a function of the elements of $\psi(\mathcal{P}_f)$ since f_2 has the $\pi\sigma$-property. Therefore there is an induced map $g_{f_2} : \mathbb{P}^{n_2} \dashrightarrow \mathbb{P}^{n_2}$. This argument is symmetric in f_1 and f_2, so the if and only if assertion holds.

Let $(\widetilde{F_1})_{\mathbf{w}}$ denote the monic polynomial associated to F_{ϕ_1}, and let $(\widetilde{F_2})_{\mathbf{w}}$ denote the monic polynomial associated to F_{ϕ_2}. Since the static portraits are isomorphic, the the polynomials $(\widetilde{F_1})_{\mathbf{w}}$ and $(\widetilde{F_2})_{\mathbf{w}}$ will have exactly the same form. They will be identical except for possible relabeling of variables, and up to normalization. Hence the algebraic degrees of the induced maps g_{f_1} and g_{f_2} must be the same. $\square$

The proposition above is very inspiring. We see that to determine if there is an induced map, we should look at the static portrait of F_ϕ. We obtain the following theorem.

Theorem 10.1.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d. Identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way, and suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If $\mathrm{rank}(R_f) < |\mathcal{P}_f|$, and if g_f is an endomorphism, then*

$$\text{alg deg}(g_f) = d.$$

Proof. We have already seen in corollary 5.2.1, that if alg $\deg(g_f) = d$, then g_f is an endomorphism. As mentioned in the proof of corollary 5.2.1, alg $\deg(g_f) \geq d$. We suppose now that alg $\deg(g_f) > d$ and then prove that the induced map g_f has points of indeterminacy. The polynomial F_ϕ has the following form:

$$F_\phi(z) = \alpha_d z^d + \ldots + \alpha_0.$$

Consider the minimal portrait for F_ϕ, the set of equations contained in eq_{F_ϕ}, and the set D_{F_ϕ}. Normalize some way so that we identify $\phi \in \mathrm{Mod}(S^2, \mathcal{P}_f)$ with the point $[x_1 : \ldots : x_n : 1] \in \mathbb{P}^n - \Delta$. Enumerate the elements of D_{F_ϕ} as

$$D_{F_\phi} = \{\omega_1, \ldots, \omega_k, x_1, \ldots, x_r\},$$

where $\{x_1, \ldots, x_r\} = D_{F_\phi} \cap \psi(\mathcal{P}_f)$. Notice that the rank of R_f is then equal to r. A priori, we can adjust our identification of $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ to label the points this way. Since f has the $\pi\sigma$-property, we can solve the equations in eq_{F_ϕ} for the α_i, writing α_i as a function of $x_1, \ldots, x_r$.

We can then consider the monic polynomial associated to F_ϕ, which we may write as

$$\widetilde{F_{\mathbf{w}}}(z) = z^d + \frac{p_{d-1}(\{w_1, \ldots, w_r, w_{r+1}\})}{q_{d-1}(\{w_1, \ldots, w_r, w_{r+1}\})} z^{d-1} + \ldots + \frac{p_0(\{w_1, \ldots, w_r, w_{r+1}\})}{q_0(\{w_1, \ldots, w_r, w_{r+1}\})}$$

where we relabel so that w_{r+1} is the extra coordinate obtained in transforming to homogeneous coordinates. Note that the p_i and q_i are homogeneous polynomials which are assumed to have no common factors. If alg deg $(g_f) > d$, then there is a q_j so that $\deg(q_j) > 0$.

So the polynomial $\widetilde{F_{\mathbf{w}}}$ induces a map $\widetilde{G_f} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ which is not holomorphic. Since $\widetilde{G_f}$ is not holomorphic, there exists a nonconstant homogeneous polynomial $p(\{w_1, \ldots, w_r, w_{r+1}\})$ of minimal degree such that the map

$$H(\mathbf{w}) := (p(\{w_1, \ldots, w_r, w_{r+1}\}) \cdot v_1(\mathbf{w}), \ldots, p(\{w_1, \ldots, w_r, w_{r+1}\}) \cdot v_{n+1}(\mathbf{w}))\,,$$

is holomorphic. Notice that $p(\{w_1, \ldots, w_r, w_{r+1}\})$ is defined up to scaling by a nonzero complex number, and only depends on the variables $w_1, \ldots, w_{r+1}$. The algebraic degree of g_f is equal to $d + \deg(p)$. Consider the polynomial

$$h_{\mathbf{w}}(z) := p(\mathbf{w}) \cdot \widetilde{F_{\mathbf{w}}}(z).$$

This polynomial is homogeneous in the variables w_i (for $i \in [1, r+1]$) and z, and it is homogeneous of degree $d + \deg(p)$. Moreover, this polynomial induces the map $H : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, and the following diagram commutes.

$$\begin{tikzcd}
\mathbb{C}^{n+1} - \{\mathbf{0}\} \arrow[r, "H"] \arrow[d] & \mathbb{C}^{n+1} \arrow[d, dashed] \\
\mathbb{P}^n \arrow[r, dashed, "g_f"] & \mathbb{P}^n
\end{tikzcd}$$

We may express the polynomial $h_{\mathbf{w}}$ as

$$h_{\mathbf{w}}(z) = p(\{w_1, \ldots, w_{r+1}\})z^d + \beta_{d-1}(\{w_1, \ldots, w_{r+1}\})z^{d-1} + \ldots + \beta_0(\{w_1, \ldots, w_{r+1}\})$$

where β_i is a homogeneous polynomial in the w_i, of degree greater than 0.

A priori, we can change our normalization to suppose that

$$\{0, \infty\} \subset \{x_1, \ldots, x_r\},$$

where $x_{r+1} = 1$. Then we see that the coefficients of the polynomial $h_{\mathbf{w}}$ depend only on $r - 2$ variables. Reindex, and suppose that

$$h_{\mathbf{w}}(z) = p(\{w_1, \ldots, w_{r-2}\})z^d + \beta_{d-1}(\{w_1, \ldots, w_{r-2}\})z^{d-1} + \ldots + \beta_0(\{w_1, \ldots, w_{r-2}\}).$$

The rank of R_f is equal to r. If $r - 2 < n + 1$, then we can manufacture a point of indeterminacy for the induced map g_f.

Suppose that $\text{rank}(R_f) < |\mathcal{P}_f| \implies r - 2 < n + 1$. Consider the locus in $\mathbb{C}^{n+1}$ defined by $Q := (0, \ldots, 0, w_{r-1}, \ldots, w_{n+1})$. Since the coefficients of $h_{\mathbf{w}}$ are homogeneous polynomials which depend only on $w_1, \ldots, w_{r-2}$, $h_{\mathbf{w}}$ must be identically 0 on Q. Therefore, if $r - 2 < n + 1$, then the locus

$$\mathcal{I} := [0 : \ldots : 0 : w_{r-1} : \ldots : w_{n+1}] \in \mathbb{P}^n$$

is part of the indeterminacy locus for the induced map g_f. $\square$

The previous proposition gives us the following corollary.

Corollary 10.1.1. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d. Identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way, and suppose that f has the $\pi\sigma$-property, such that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If $\mathrm{rank}(R_f) = 2$, or if $\mathrm{rank}(R_f) = 3$, then g_f is necessarily an endomorphism which is postcritically finite.*

Proof. We prove that the algebraic degree of any such g_f must be equal to d. Find a Thurston map f' such that $|\mathcal{P}_{f'}| = 4$ and $\mathrm{Min}(F_\phi)$ is isomorphic to $\mathrm{Min}(F'_\phi)$. By proposition 10.1.1, there is an induced map $g_{f'} : \mathbb{P}^1 \to \mathbb{P}^1$, such that

$$\text{alg deg } (g_f) = \text{alg deg } (g_{f'}).$$

Since $\mathrm{Min}(F_\phi)$ and $\mathrm{Min}(F'_\phi)$ are isomorphic, $\mathrm{rank}(R_f) = \mathrm{rank}(R_{f'})$, so we see that the condition $\mathrm{rank}(R_{f'}) < |\mathcal{P}_{f'}| = 4$, is automatically satisfied. Hence, $g_{f'} : \mathbb{P}^1 \to \mathbb{P}^1$ is holomorphic if and only if alg deg $(g_{f'}) = d$. But every such map is holomorphic on $\mathbb{P}^1$, so we must always have the condition that alg deg $(g_{f'}) = d$. Hence alg deg $(g_f) = d$ as well, so g_f is a postcritically finite endomorphism. □

We summarize the necessary and sufficient conditions for the induced map g_f to be holomorphic with theorem 10.1.2.

Theorem 10.1.2. *Let the Thurston map $f : S^2 \to S^2$ be a topological polynomial of degree d. Identify $\mathrm{Mod}(S^2, \mathcal{P}_f)$ with $\mathbb{P}^n - \Delta$ in the standard way, and suppose that f has the $\pi\sigma$-property, and that F_ϕ induces a map $g_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If $d < |\mathcal{P}_f| - 3$, or if $\mathrm{rank}(R_f) < |\mathcal{P}_f|$, then g_f is an endomorphism if and only if*

$$\text{alg deg}(g_f) = d.$$

CHAPTER 11

IN SUMMARY

In this thesis we constructed postcritically finite endomorphisms of $\mathbb{P}^n$ using the combinatorics of various Thurston maps in chapter 4. We defined the key idea behind the construction: the $\pi\sigma$-property of a Thurston map f in chapter 5. We established some results about the necessary and sufficient conditions under which the maps g_f are holomorphic in chapters 5 and 10, and we interpreted the periodic cycles of g_f in $\mathbb{P}^n - \Delta$ in chapter 8.

In our analysis, we established a link between the induced map $g_f : \mathbb{P}^n \to \mathbb{P}^n$, and the pullback map $\sigma_f : \text{Teich}(S^2, \mathcal{P}_f) \longrightarrow \text{Teich}(S^2, \mathcal{P}_f)$. We were able to exploit this link to deduce new results about σ_f in chapter 6, and results about g_f in chapters 7, 8, and 9. We hope to continue to exploit this connection between the maps to learn even more about the complex dynamics and Teichmüller theory involved.

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